A unified physical presentation of mixed, mixed-hybrid finite elements and usual finite differences for the determination of velocities in waterflow problems

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Abstract : The objective of this paper is to give a non-mathematical presentation of mixed finite elements, following as much as possible physical intuition and natural good sense. Mixed and mixed hybrid finite elements are clearly presented as two formulations of the same approximation, and the use of a quadrature rule is shown to reduce both formulations to the usual finite difference scheme. Application to the resolution of stationary (elliptic) and time dependent (parabolic) water flow problems is discussed in detail for the case of a uniform rectangular grid, and the enhancement in the approximation of velocities is emphasized. Generalizations and order of convergence are discussed.

UNE PRESENTATION PHYSIQUE UNIFIEE DES ELEMENTS FINIS MIXTES, MIXTES HYBRIDES ET DES DIFFERENCES FINIES USUELLES POUR LA DETERMINATION DES VITESSES DANS LES ECOULEMENTS D'EAU EN MILIEUX POREUX.

Résumé : L'objectif de cet article est de donner une présentation non-mathématique des éléments finis mixtes en suivant autant qu'il est possible l'intuition physique et le bon sens. Les éléments finis mixtes et mixtes hybrides sont ainsi présentés comme deux avatars de la même approximation, qui d'ailleurs redonne le schéma habituel des différences finies lorsqu'on utilise une formule de quadrature numérique adéquate. On discute en détail l'application des ces techniques au problème stationnaire (elliptique) et d'évolution (parabolique) d'écoulements d'eau en milieux poreux dans le cas d'un maillage rectangulaire uniforme, ainsi que l'amélioration apportée à l'approximation des champs de vitesse. Les généralisations possibles ainsi que les ordres de convergence sont discutés.

Key words

Mixed finite elements, mixed hybrid finite elements, approximation of velocities, waterflow problems, aquifers.

Mots clefs

Eléments finis mixtes, éléments finis mixtes hybrides, approximation des vitesses, problèmes d'écoulements d'eau, aquifères.

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CONTENT

1. Introduction

The need for more precise simulation of the transport of pollutants by underground water has drawn the attention of hydrologists to a class of approximation techniques known under the generic name of Mixed Finite Element Methods. The basic idea here is to approximate simultaneously the pressure P and its gradient, or more generally a gradient related velocity field $\vec{q}$ in such a way that both P_h and $\vec{q}_h$ can be proven to converge, in adequate norms, to their continuous counterparts, and that the approximated velocity field $\vec{q}_h$ retains one important feature of the exact velocity field $\vec{q}$ namely that it is continuous on each element and has a continuous normal component when passing from one element to the other. This makes a velocity field $\vec{q}_h$ computed using a mixed approximation a nice candidate to plug into the equation governing the transport of the pollutant, as it is more precise than the velocities calculated by finite differencing of the pressure, is naturally defined everywhere where it is needed, and allows for an unambiguous (and even exact in some cases) determination of the trajectories of particules.

The idea for mixed finite elements comes from structural mechanics, cf. [13] where it was introduced in 1967. Since, mixed methods have been developped for a wide variety of domains including Stokes' problem, cf. [12], flow in porous media cf. [6], and electromagnetism, cf.[2] [14]. The method presented here is based on that given in [15],[17] where the fundamental approximation spaces, the Raviart-Thomas spaces were introduced.

One might wonder why the idea of mixed methods has not been more frequently exploited in many engineering domains. There may be at least three explanations for this :

first, in its original setting, the mixed approximation led, for stationary problems, to the resolution of a symmetric positive semidefinite system in both unknowns P_h and $\vec{q}_h$ whose solution was quite delicate and time consumming, which deterred potentiel users.

second, it's only recently [1] that the popularization of the mixed-hybrid formulation introduced originally in mechanical engineering cf.[11] reduced the calculation of a mixed approximation to the solution of a classical symmetrical positive definite system with twice as many unknowns (in 2D) as classical finite differences.

third, the mathematical theory of mixed finite elements is quite involved, and much of the literature on mixed and mixed hybrid finite elements gives a mathematically rigorous presentation of these elements or a treatment highly specialized for engineering applications which in either case is difficult to read by people who have not been specially trained.

Hence the (maybe too ambitious ...) objective of this paper is to give a non-mathematical presentation of mixed finite elements, following as much as possible physical intuition and natural good sense. All calculations are elementary and abundantly detailed, and the final systems of equations corresponding to each case are clearly posted.

We have chosen to present the mixed finite elements in the case of a regular rectangular mesh, where all calculations can be carried out by hand. After describing in paragraph 2 the Raviart -Thomas space in the most elementary case, we devote paragraph 3 to the study of the consistency relations that must be satisfied by the finite dimensional pressure and velocity fields P_h and $\vec{q}_h$ in order that they have a chance of being an approximation of some function P and of its associated velocity field $\vec{q} = - a \nabla P$. We describe then in paragraph 4 the mixed and mixed hybrid approximations of stationary (elliptic) and transient (parabolic) problems; in fact we first define in paragraph 4.2, using physical intuition, a set of discrete equations, which we call the "finite element" (in its original, historical sense) approximation. We then define the mixed approximation (in paragraph 4.4.) and mixed-hybrid approximation (in paragraph 4.5) as two alternative ways of formulating the solution of the same set of discrete equations, which clearly establishes that mixed and mixed-hybrid approximations are in fact one and only one approximation ! We compare the resulting sets of equations for both the stationary and transient problems.

We point out in paragraph 5 that the above set of equations reduces, on a rectangular mesh, to the usual five star finite difference scheme when a low-order quadrature formula is used to calculate integrals on each element. An interesting by-product of this is a simple way of associating to any finite difference solution a velocity field $\vec{q}_h$ of mixed type.

We indicate in paragraph 6 the possible genralization to 3 D, irregular meshes or higher order elements, and indicate in paragraph 7 some mathematical results concerning the order of convergence of the schemes.

We hope that this presentation will give to the hydrologist reader an intuitive feeling for what mixed approximations are, help him to identify among the many variants the one the most adapted to his problem, and encourage him to put it to work on a computer!

2. The Raviart-Thomas space over one element K

We consider the square elements $K = [0,h] \times [0,h]$ with edges A_i, $i = 1,\ldots,4$ parallel to the axes.

The (lowest order) Raviart-Thomas space of vector functions over K is a finite dimensional subspace $\vec{X}_K$ of.

$$(2.1) \qquad H(\mathrm{div}, K) = \left\{ \{\vec{q}\} \in (L^2(K))^2 \mid \nabla \vec{q} \in L^2(K) \right\}$$

having the following properties :

(2.2) $\quad \forall \vec{q} \in \vec{X}_K$, $\nabla \vec{q}$ is constant over K.

(2.3) $\quad \forall i$, $i = 1,\ldots,4$, $\vec{q}.\vec{v}_K$ is constant over the edge A_i.

(2.4) $\quad$ any vector field $\vec{q} \in \vec{X}_K$ is perfectly determined by the knowledge of its flux Q_i through the edges A_i, $i = 1, \ldots, 4$.

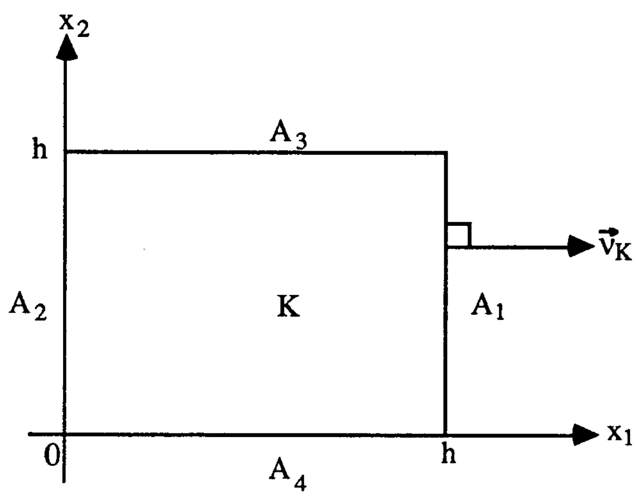


Figure 2.1 The finite element K, its edges A_i and its unit exterior normal $\vec{v}_K$.

It is then natural to use as basis functions for $\vec{X}_K$ the vector fields $\vec{w}_j$ defined by :

$$(2.5) \qquad \int_{A_i} \vec{w}_j(x) \, . \, \vec{v}_K = \delta_{ij}.$$

Hence w_j is the vector field having a flux of one through the edge A_j and a zero flux through all the remaining edges. With this basis we can write any vector $\vec{q} \in \vec{X}_K$ as:

$$(2.6) \qquad \vec{q}(x) = \sum_{j=1}^{4} Q_{K,j} \, \vec{w}_j(x)$$

where the four degrees of freedom $Q_{K,j}$ are simply the fluxes of $\vec{q}$ through all four edges of K.

It is important to notice that the fact that the normal component of $\vec{q}$ is constant along edges will allow us to construct, when combining many elements into a domain Ω, a finite dimensional space of vector fields over Ω having a continuous normal componant at the interface between two elements.

On the square element K considered here, the analytical expressions for the basis functions $\vec{w}_j$ are very simple :

$$(2.7)\qquad \vec{w}_1(x) = \begin{bmatrix} x_1 / h^2 \\ 0 \end{bmatrix} \qquad \vec{w}_2(x) = \begin{bmatrix} (x_1 - h)/ h^2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3(x) = \begin{bmatrix} 0 \\ x_2 / h^2 \end{bmatrix} \qquad \vec{w}_4(x) = \begin{bmatrix} 0 \\ (x_2 - h)/h^2 \end{bmatrix}.$$

We shall make use in the sequel of the scalar product in $L^2(K ; \mathbb{R}^2)$ of the basis functions :

$$(2.8)\qquad A_{ij} = \int_K \vec{w}_j(x)\, \vec{w}_i(x)\, dx.$$

We can then define the elementary matrix A_K associated to the element K by:

$$(2.9)\qquad A_K = [A_{i,j}] = \frac{1}{6} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

whose inverse is :

$$(2.10)\qquad A_K^{-1} = 2 \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Finally, we see from (2.5.) that :

$$(2.11)\qquad \int_K \nabla \vec{w}_j(x) = \int_{\partial K} \vec{w}_j . \vec{\nu}_K = 1$$

and hence :

$$(2.12)\qquad \int_K \nabla \vec{q}(x)\, dx = \sum_{j=1}^{4} Q_{K,j}$$

3. Approximating a function and its gradient : the Q_1, mixed, and mixed-hybrid finite elements

We are given now :

a domain $\Omega = [0, L_1] \times [0, L_2]$ of $\mathbb{R}^2$

a function $a : \Omega \rightarrow \mathbb{R}^+$

and we are going to investigate various ways of approximating a pressure field P over Ω and the associated velocity field $\vec{q} = - a \nabla P$, when Ω is covered by a uniform grid made of $N_1 N_2$ square elements K of size h (see figure 3.1).

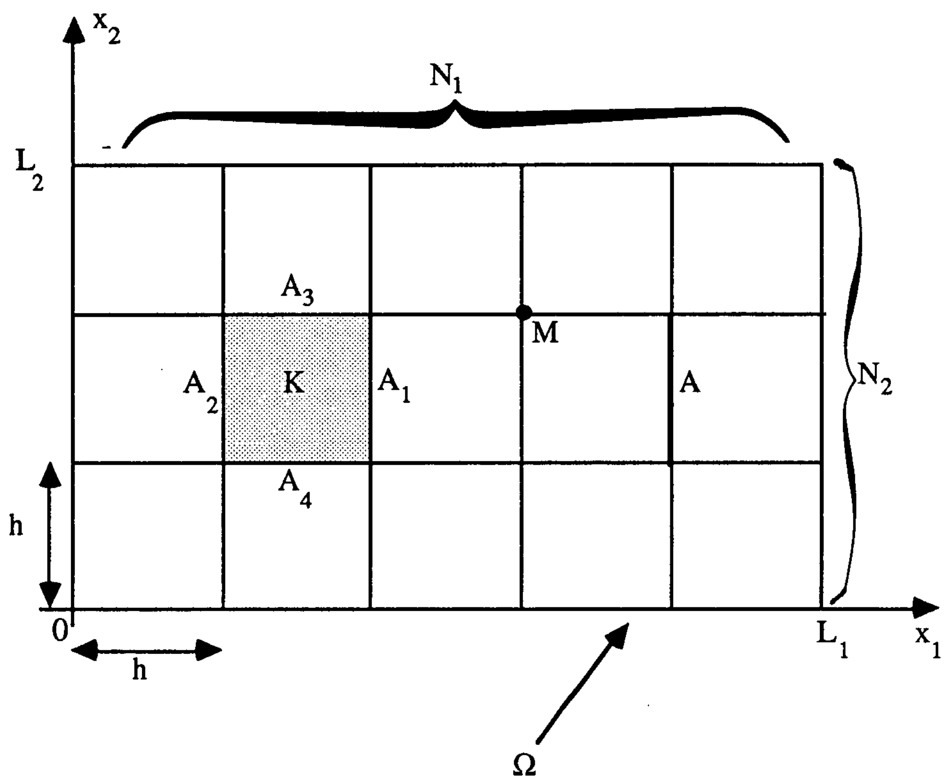


Figure 3.1: The domain Ω and its discretization.

We suppose that P and $\vec{q}$, as functions of the space variable x, have the usual degree of regularity of the variational solutions of elliptic or parabolic problems :

$$(3.2) \qquad P \in V = H^1(\Omega) = \left\{ P \in L^2(\Omega) \, ; \, \frac{\partial P}{\partial x_i} \in L^2(\Omega), i = 1,2 \right\}$$

$$(3.3) \qquad \vec{q} \in \vec{X} = H(\mathrm{div}, \Omega) = \left\{ \vec{q} \in (L^2(\Omega))^2 ; \; \nabla \vec{q} \in L^2(\Omega) \right\}$$

We shall use throughout this paper the notation :

$$(3.4) \qquad \begin{aligned} \mathcal{T}_h &= \{\text{mesh elements } K\} \\ \mathcal{A}_h &= \{\text{mesh edges } A\} \\ \mathcal{S}_h &= \{\text{mesh vertices } M\} \end{aligned}$$

and shall denote by $a_h(x)$ a piecewise constant function approximating the function $a(x)$:

$$(3.5) \qquad a_h(x) = \text{constant} = a_K \quad \forall x \in K, \forall K \in \mathcal{T}_h$$

• <u>Classical Q_1 finite element approximation.</u>

<u>The approximation here concerns only the pressure P :</u>

$$(3.6) \qquad \begin{cases} \text{on each element K one approximates P by} \\ P_K \in Q_1 \ (K) = \{\text{bilinear polynomials in } x_1 \text{ and } x_2 \end{cases}$$

Then P_K is perfectly defined by its four values at the vertices of K. By juxtaposing the elements K over Ω, we obtain as an approximation of P, a function P_h which is continuous on Ω and Q_1 on each element, and whose degrees of freedom are the (N_1+1) (N_2+1) values on the vertices of $\mathcal{S}_h$.

This classical approximation leads, for an elliptical problem, to a symmetrical positive definite linear system with Card $\mathcal{S}_h = (N_1+1)(N_2+1)$ unknowns, which is very easy to solve; this accounts for the success of this approximation. However the situation is not so bright if one is also interested in having an approximation of the velocity field $\vec{q}$. Of course, nothing prevents us from defining, as P_h is a continuous piecewise bilinear function :

$$\vec{q_h}\,(x) = -\,a_h(x)\,\nabla P_h(x),$$

which defines $\vec{q}_h\,(x)$ at every interior point x of the element K. But this velocity field is discontinuous at interfaces between elements, so that it does not allow for a precise calculation of the flux through an interior edge (this flux may have two values depending on which side of the edge one computes it) and hence does not give a mass preserving scheme. Similarly, if one tries to propagate a particule using the velocity field $\vec{q}$ one may be in trouble when, on each side of an edge A, the component of $\vec{q}_h$ normal to the edge drives the particle towards the edge. (This may happen as the normal component of $\vec{q}_h$ may be discontinuous across the edge!)

Hence the Q_1 finite element approximation is not suited for the determination of velocity fields.

•The mixed and mixed-hybrid finite element approximations.

We shall present in this paragraph these two approximations in a single mold, postponing until paragraph 4.3 the precise definitions of the mixed and mixed hybrid avatars of the basic mixed idea which consists in approximating simultaneously the pressure $P \in V = H^1(\Omega)$ and the velocity field $\vec{q} \in \vec{X} = H(\mathrm{div},\Omega)$:

$$(3.7)\quad \begin{cases} \text{On each element K we approximate P and } \vec{q} \text{ by :} & \\ \text{i)}\quad P_K \in \mathbb{R} & = \text{approximation of the mean of P on K} \\ \text{ii)}\quad TP_{K,i} \in \mathbb{R},\quad i=1,\dots,4 & = \text{approximation of the mean of P on } A_i \\ \text{iii)}\quad \vec{q}_K \in \vec{X}_K & = \text{approximation of } \vec{q} = -a\nabla P \text{ on K} \end{cases}$$

where $\vec{X}_K$ is the Raviart-Thomas space on K described in paragraph 2. As we have seen that $\vec{q}_K$ was perfectly known once its fluxes through the four edges of K are known, the approximation of P and $\vec{q}$ on K is completely determined when one knows the nine degrees of freedom :

$$(3.8)\quad \begin{cases} P_K \in \mathbb{R} & \\ TP_{K,i} \in \mathbb{R} & i = 1 \dots 4 \\ Q_{K,i} \in \mathbb{R} & i = 1 \dots 4 \end{cases}$$

These nine numbers cannot be chosen completely arbitrarily, as the quantities P and $\vec{q}$ which they serve to approximate are related by :

$$(3.9)\qquad \vec{q} = -a\,\nabla P\cdot$$

In order to find the consistency relation which we shall impose on the unknowns (3.8) on K, we begin by writting (3.9) in a variational form over the element K. Dividing by a yields :

$$(3.10)\qquad \frac{\vec{q}}{a} = -\nabla P\cdot$$

Then taking the scalar product of (3.10) with a test function $\vec{s} \in H(\mathrm{div}, K)$, integrating over K and using a Green's formula we obtain

$$(3.11)\qquad \int_K \frac{\vec{q}}{a}\cdot\vec{s} = \int_K P\ \nabla\vec{s} - \int_{\partial K} P\ \vec{s}\cdot\vec{\nu}_K \quad \forall\, \vec{s} \in H(\mathrm{div}, K).$$

The sought consistency equation for the approximate quantities $\vec{q}_K$, P_K (over K) and TP_K (over ∂K) will then be obtained by requiring that they satisfy a relation similar to (3.11), namely :

$$(3.12)\qquad \int_K \frac{\vec{q}_K \cdot s_K}{a_K} = \int_K P_K \nabla \vec{s}_K - \sum_{j=1}^{4} \int_{A_j} TP_{K,j}\, \vec{S}_K \cdot \vec{v}_K \qquad \forall\, \vec{s}_K \in \vec{X}_K$$

Taking for test function $\vec{s}_K$ successively the four basis functions $\vec{w}_i$ of $\vec{X}_K$ defined by (2.5) yields, as a_K and P_K are constant over K and $TP_{K,j}$ is constant on the edge A_j :

$$\frac{1}{a_K} \sum_{j=1}^{4} Q_{K,j} \int_K \vec{w}_j \cdot \vec{w}_i = P_K \int_K \nabla \vec{w}_i - \sum_{j=1}^{4} TP_{K,j} \int_{A_j} \vec{w}_i \cdot \vec{v}_K$$

or, using the notation (2.8) and formulas (2.11) and (2.5)

$$(3.13)\qquad \frac{1}{a_K} \sum_{j=1}^{4} A_{i,j}\, Q_{K,j} = P_K - TP_{K,i} \quad i = 1,2\ldots 4,$$

or in matrix notation :

$$\boxed{(3.14)\qquad A_K\, Q_K = a_K\,(P_K\, DIV_K^T - TP_K)}$$

where A_K is the elementary 4 x 4 matrix defined in (2.9), and where

$$(3.15)\qquad DIV_K = [\,1 \;\; 1 \;\; 1 \;\; 1\,] \text{ is the elementary divergence matrix on K,}$$

and where

$$(3.16)\qquad DIV_K^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\; Q_K = \begin{bmatrix} Q_{K,1} \\ Q_{K,2} \\ Q_{K,3} \\ Q_{K,4} \end{bmatrix},\; TP_K = \begin{bmatrix} TP_{K,1} \\ TP_{K,2} \\ TP_{K,3} \\ TP_{K,4} \end{bmatrix}.$$

As we have seen in (2.10) that the matrix A_K is invertible, equation (3.14) allows us to calculate the vector Q_K (and hence the velocity field $\vec{q}_K$!) as soon as we know the mean element pressure P_K and the four mean edge pressures TP_{Ki} $i = 1,\ldots,4$. It is thus the exact finite dimensional counterpart of the equation, $\vec{q} = - a\,\nabla P$ and will be called the consistency equation.

We now have to define P_h, TP_h and $\vec{q}_h$ in such a way that they may be used for approximating P and $\vec{q}$ in a consistent way over all Ω. So let define :

$$(3.17) \qquad P_h : K \in \mathcal{T}_h \to P_K \in \mathbb{R}$$

$$(3.18) \qquad TP_h : A \in \mathcal{A}_h \to \{TP_{K,A} \in \mathbb{R}, \ \forall K \in \mathcal{T}_h, \partial K \supset A\}$$

$$(3.19) \qquad \vec{q}_h : K \in \mathcal{T}_h \to \vec{q}_K \in \vec{X}_K.$$

We first see that P_h, TP_h and $\vec{q}_h$ are simply obtained by juxtaposing the approximated quantities P_K, $TP_{K,i}$ and $\vec{q}_K$ relative to each element K of the mesh $\mathcal{T}_h$. But one can also think of P_h, TP_h and $\vec{q}_h$ as being functions :

$$(3.20) \qquad \forall x \in \Omega, \ P_h(x) = P_K$$

where K is the element of $\mathcal{T}_h$ containing x.

$$(3.21) \qquad \forall x \in A_h, \ TP_h(x) = \begin{cases} TP_{K,A} & \text{if } x \in A = \text{boundary edge} \\ \{TP_{K,A}, TP_{K',A}\} & \text{if } x \in A = \text{interior edge} \end{cases}$$

$$(3.22) \qquad \forall x \in \Omega, \ \vec{q}_h(x) = \vec{q}_K(x)$$

where K is the element of $\mathcal{T}_h$ containing x.

Now, we have, on each interior edge A, two values of the pressure ($TP_{K,A}$ and $TP_{K',A}$) and of the flux ($Q_{K,A}$ and $Q_{K',A}$).

Let us now write down the relations that must be satisfied by P_h, TP_h and $\vec{q}_h$ in order to have a consistent approximation, over the whole of Ω, of $P \in H^1(\Omega)$ and $\vec{q} \in H(\mathrm{div},\Omega)$

- on each element K, the approximation has to be consistent with $\vec{q} = -a\nabla P$ and hence must satisfy (3.14), ie :

$$(3.23) \qquad A_K Q_K = a_K\left(P_K \, DIV_K^T - TP_K\right) \forall K \in \mathcal{T}_h$$

(consistency equation)

- the function P to be approximated being in the Sobolev space $H^1(\Omega)$, must have a uniquely defined value (trace) on each edge $A \in \mathcal{A}_h$. We shall thus require that:

(3.24) $TP_{K,A} = TP_{K',A}$ $\forall$ $A \in \mathcal{A}_h$, A interior

(continuity of pressures)

• the velocity field $\vec{q}$ to be approximated being in the space H(div,Ω), the value (trace) of its normal component on each interior edge A is uniquely defined. We shall thus require that :

(3.25) $Q_{K,A} + Q_{K',A} = 0$ $\forall$ $A \in \mathcal{A}_h$, A interior

(continuity of fluxes)

Recalling the properties of the Raviart-Thomas space $\vec{X}_K$ we see that equation (3.25) will be sufficient to insure continuity of the normal component of $\vec{q_h}$ at every point of any interior edge.

Summarizing, we see that (3.17 through 3.19) yields a consistent approximation of P and $\vec{q} = - a \nabla P$ on Ω as soon as equations (3.23 through 3.25) are satisfied.

Considering now the mesh of figure 3.1, we see that this approximation has 9 N_1 N_2 unknowns (9 per element !). As the conditions (3.23 through 3.25) yield 8 N_1 N_2 - 2 (N_1 N_2) equations, there are exactly N_1 N_2 + 2(N_1 + N_2) equations missing in order to have the same number of equations as unknowns, i.e. :

(3.26.) one equation per element $K \in \mathcal{T}_h$, i.e. N_1 N_2,

(3.27) one equation per boundary edge $A \in \mathcal{A}_h$, ie 2(N_1 + N_2).

These missing equations will be introduced in paragraph 4, and will consist of the balance equation corresponding to the elliptic or parabolic equation to be solved (for (3.26)) and of the corresponding boundary conditions (for (3.27)).

4. Finite element approximations of elliptic and parabolic problems

4.1 The continuous problem

Let

(4.1) $T > 0$

be given. We consider, on the rectangular set Ω of figure 3.1, the following evolutionary problem from time 0 to time T :

(4.2) $c(x)\,\dfrac{\partial P}{\partial t} - \nabla(a(x)\,\nabla P) = f(x,t)$ on Ω for $t \in\ ]0,T[$

(4.3) $P = P_e$ on Γ_D for $t \in\ ]0,T[$

(4.4) $-a\,\dfrac{\partial P}{\partial \nu_\Omega} = q_e$ on Γ_N for $t \in\ ]0,T[$

(4.5) $P(x,0) = P_0(x)$ on Ω for $t = 0$,

where (see figure 4.1)

(4.6) $\begin{cases} \Gamma = \text{boundary of } \Omega,\ \nu_\Omega \text{ is the exterior normal to } \Omega \\ \Gamma_D = \text{part of } \Gamma \text{ with given pressure condition} \\ \Gamma_N = \text{part of } \Gamma \text{ with given flow condition} \end{cases}$

and :

(4.7) $\begin{cases} P_e : \Gamma_D \to \mathbb{R} \text{ is a given pressure profile on } \Gamma_D \\ q_e : \Gamma_N \to \mathbb{R} \text{ is a given flow profile on } \Gamma_N \\ P_0 : \Omega \to \mathbb{R} \text{ is a given initial pressure profile on } \Omega \\ f : \Omega \times]0\ T[\to \mathbb{R} \text{ represents the source or sink terms} \end{cases}$

(4.8) $\begin{cases} a : \Omega \to \mathbb{R}^+ \text{ is the transmissivity coefficient.} \\ c : \Omega \to \mathbb{R}^+ \text{ is the storage coefficient.} \end{cases}$

We shall make the following hypothesis concerning a and c

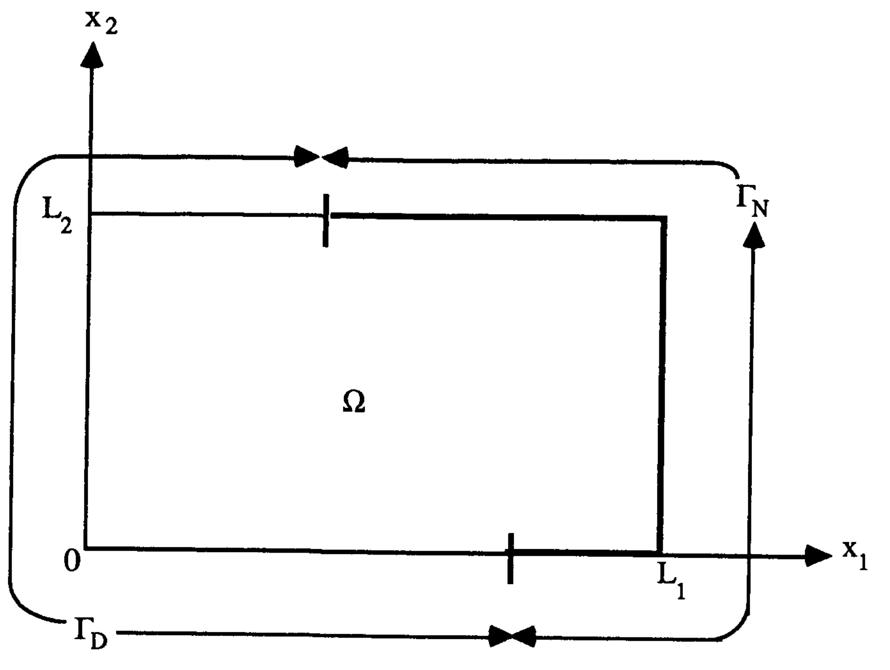


Figure 4.1 The domain Ω and the partition of its boundary

(4.9) $\quad a(x) \geq a_{min} > 0$ for all $x \in \mathbb{R}$

and either

(4.10) $\quad c(x) \geq c_{min} > 0$ for all $x \in \Omega$

which implies that (4.2 through 4.5) is a parabolic equation, or

(4.11) $\quad c(x) = 0 \qquad$ for all $x \in \Omega$

which implies that (4.2 through 4.4) is a family of elliptic equations (in which case the inital condition (4.5) of course becomes useless).

4.2 Finite element approximation of the continuous problem.

Let now Ω be covered by a rectangular grid as in paragraph 3. We suppose that this grid is adapted to the boundary conditions in the sense that (see figures 4.1 and 4.2) :

$$(4.12) \quad \begin{cases} \Gamma_D \text{ is made up of a finite number of boundary edges A of } \mathcal{A}_h. \\ \Gamma_N \text{ is made up of a finite number of boundary edges A of } \mathcal{A}_h \end{cases}$$

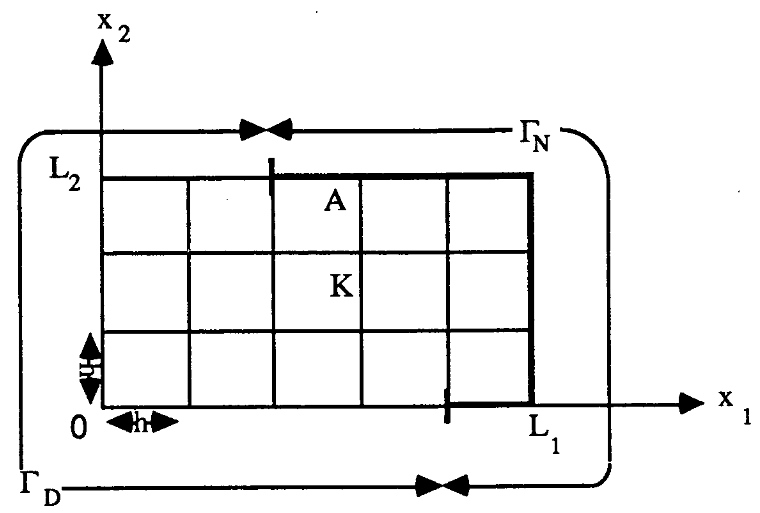


Figure 4.2 The domain Ω, its discretization and boundary conditions

Let also the time interval $]0,T[$ be discretized into N intervals of length Δt :

$$(4.13)\qquad \begin{aligned} t^n &= n\nabla t \qquad\qquad n = 0,1,\ldots,N \\ \Delta t &= T/N \end{aligned}$$

We begin by defining discretized functions approximating the right-hand sides and coefficients of equations (4.2) through (4.5) :

$$(4.14)\quad \begin{cases} \text{On each element } K \in \mathcal{T}_h \text{ we define (for example using the mean value) :} \\ F_K^n \in \mathbb{R} \quad = \text{approximation of } \int_K f(x,t^n)\,dx \\ P_{0,K} \in \mathbb{R} \quad = \text{approximation of } P_0(x) \\ c_K \in \mathbb{R} \quad = \text{approximation of } c(x) \\ a_K \in \mathbb{R} \quad = \text{approximation of } a(x) \end{cases}$$

and :

$$(4.15)\quad \begin{cases} \text{On each edge } A \subset \Gamma_D \text{ we define (also possibly using the mean value):} \\ P_{e,A} = \text{approximation of } P_e(x) \text{ on } A \end{cases}$$

$$(4.16)\quad \begin{cases} \text{On each edge } A \subset \Gamma_N \text{ we define} \\ Q_{e,A} = \text{approximation of the flux imposed through A, i.e. of } \int_A q_e(x)\,dx. \end{cases}$$

We turn now to the approximation of equations (4.2) through (4.5) which we shall perform using a finite element approach in its original mechanical setting, where the equilibrium equation was approximated separately on each finite element K, and continuity of displacements and constraints were imposed at junctions between elements in order to obtain the global solution of the problem over Ω.

As it is well known that solving equations (4.2) through (4.5) is rigously equivalent to satisfying :

i) equation (4.2) on every element K of $\mathcal{T}_h$,

ii) continuity of the pressure and of the normal component of velocity on all interior edges,

iii) boundary condition (4.3) on all edges $A \subset \Gamma_D$, and boundary condition (4.4) on all edges $A \subset \Gamma_N$,

iv) initial condition (4.5) on every element K of $\mathcal{T}_h$.

we shall construct our finite element approximation to (4.2) through (4.5) by:

i) Approximating equation (4.2) separately on each finite element K using the mixed-hybrid approximation (3.7) $P_K \in \mathbb{R}$, $TP_{K,i}, \in \mathbb{R}, i = 1, \dots, 4$ and $\vec{q}_K \in \vec{X}_K$.

As we have already seen in paragraph 3, these quantities must satisfy

$$(3.23) \qquad A_K Q_K^n = a_K \left(P_K^n DIV_K^T - TP_K^n\right) \forall K \in \mathcal{T}_h \; \forall n = 0,1 \dots N$$

(consistency equation)

This equation has to be completed by one other expressing the fact that P_K, TP_K, and $\vec{q}_K$ satisfy equation (4.2) in some approximate sense. For this purpose, we first rewrite (4.2) using the $\vec{q}$ unknown :

$$(4.17) \qquad c(x) \frac{\partial p}{\partial t} + \nabla \vec{q} = f(x,t) \text{ on K for } t \in \,]0,T[$$

which when multiplied by a test function $v \in L^2(K)$ and integrated over K yields the variational formulation of (4.2) over K :

$$(4.18) \qquad \int_K c(x) \frac{\partial P}{\partial t} v + \int_K v \nabla \vec{q} = \int_K f(x,t) v \qquad \forall v \in L^2(K).$$

As the chosen approximations P_K and $\vec{q}_K$ satisfy :

P_K = constant over K (cf.(3.7))

$$\nabla \vec{q_K} = \frac{1}{|K|} \sum_{A \subset \partial K} Q_{K,A} = \text{constant over } K \text{ (cf. (2.2) and (2.12))}$$

it will be natural to require of P_K and $\vec{q}_K$ that they satisfy an equation analoguous to (4.18.) for all test functions v_K = constant over K :

$$\text{(4.18bis)} \qquad \int_K \frac{P_K^n - P_K^{n-1}}{\Delta t} v_K + \int_K v_K \nabla q_K^{n(\theta)} = \int_K f^{n(\theta)} v_K \qquad \forall v_K \in \mathbb{R},$$

which then may be written, after division by v_K and using (2.12) and (4.14), as the <u>balance equation :</u>

$$\text{(4.19)} \qquad |K| \, c_K \frac{P_K^n - P_K^{n-1}}{\Delta t} + \sum_{A \subset \partial K} Q_{K,A}^{n(\theta)} = F_K^{n(\theta)}$$

where :

$$\text{(4.20)} \qquad \theta \in [0,1]$$

is the parameter of the θ – method used for the time discretization, and where :

$$\text{(4.21)} \qquad Q_{K,A}^{n(\theta)} = (1-\theta)\, Q_{K,A}^{n-1} + \theta\, Q_{K,A}^{n}$$

$$\text{(4.22)} \qquad F_K^{n(\theta)} = (1-\theta)\, F_K^{n-1} + \theta\, F_K^{n}$$

(4.23) $|K| = h^2$ is the area of the element K.

Using the matrix notation of (3.15) (3.16), the equation (4.19) may be written

$$\boxed{\text{(4.24)} \quad \left\{ \begin{array}{l} |K| \, c_K \dfrac{P_K^n - P_K^{n-1}}{\Delta t} + DIV_K \; Q_K^{n\,(\theta)} = F_K^{n\,(\theta)} \\ \qquad\qquad \forall K \in \mathcal{T}_h, \forall n = 1,2, \dots, N \end{array} \right. \quad \text{(balance equation)}}$$

Notice that this is the "equation per element K of $\mathcal{T}_h$" which was sought after in (3.26) when we were counting the number of unknowns and equations.

ii) <u>Ensuring continuity of pressures and normal components of velocities across interior edges.</u>

As we have seen at the end of paragraph 3, this can be taken care of be writting :

$$TP^n_{K,A} = TP^n_{K',A} \quad \forall A \text{ interior}, \forall n = 0,1,\ldots,N \qquad (3.24)$$

(continuity of pressures)

and :

$$Q^n_{K,A} + Q^n_{K',A'} = 0 \ \forall A \text{ interior}, \forall n = 0,1,\ldots,N \qquad (3.25)$$

(continuity of fluxes)

Notice that equation (3.25) ensures that the approximate velocity field $\vec{q_h}$ (with the notation (3.19)) has its normal component exactly continuous at every point of any interior edge A, whereas equations (3.24) yield only an approximation to pressure continuity, as the element value P_K and the edge value $TP_{K,A}$ are usually distinct.

iii) Approximating the boundary conditions (4.3) and (4.4), which, in view of definitions (4.15) and (4.16) reduces to :

$$TP^n_{K,A} = P^n_{e,A} \quad \forall A \subset \Gamma_D, \forall n = 0,1,\ldots,N \qquad (4.25)$$

(pressure boundary conditions)

$$Q^n_{K,A} = Q^n_{e,A} \ \forall A \subset \Gamma_N, \forall n = 0,1\ldots,N \qquad (4.26)$$

(flux boundary conditions)

Equations (4.25) and (4.26) are the "one equation per boundary edge "sought after in (3.27). As we saw then, we have now exactly as many equations as unknowns at each time step to determine (with the notation 3.17 through 3.19) , P^n_h, TP^n_h and $\vec{q}^n_h$ from P^{n-1}_h, TP^{n-1}_h and $\vec{q}^{n-1}_h$.

iv) Approximating the inital condition (4.5), which in view of (4.14) reduces to :

$$P^0_K = P_{0,K} \qquad \forall K \in \mathcal{T}_h \qquad (4.27)$$

(initial condition)

4.3. Summary of discrete equations

We sumarize here the equations for the

Finite Element Approximation of (4.2) through (4.5)

(4.28) $$\begin{cases} A_K Q_K^n = a_K \left(P_K^n \, DIV_K^T - TP_K^n\right) \\ \forall\, K \in \mathcal{T}_h,\ \forall n = 0,1,\ldots,N \\ \qquad \text{(consistency equation)} \end{cases}$$

(4.29) $$\begin{cases} |K|\, c_K \dfrac{P_K^n - P_K^{n-1}}{\Delta t} + DIV_K\, Q_K^{n(\theta)} = F_K^{n\,(\theta)} \\ \forall K \in \mathcal{T}_h\ ,\ \forall n = 1,2,\ldots,N \\ \qquad \text{(balance equation)} \end{cases}$$

(4.30) $$\begin{cases} TP_{K,A}^n = TP_{K',A}^n \\ \forall\ A \in \mathcal{A}_h,\ A \text{ interior},\ \forall\ n = 0,1,\ldots,N \\ \qquad \text{(continuity of pressures)} \end{cases}$$

(4.31) $$\begin{cases} Q_{K,A}^n + Q_{K',A}^n = 0 \\ \forall\ A \in \mathcal{A}_h,\ A \text{ interior},\ \forall n = 0,1\ldots,\ N \\ \qquad \text{(continuity of fluxes)} \end{cases}$$

(4.32) $$\begin{cases} TP_{K,A}^n = Pe_A^n \\ \forall\ A \in \mathcal{A}_h,\ A \subset \Gamma_D,\ \forall n = 0,1,\ldots\ ,N \\ \qquad \text{(pressure boundary condition)} \end{cases}$$

(4.33) $$\begin{cases} Q_{K,A}^n = Q_{e,A}^n \\ \forall\ A \in A_h,\ A \subset \Gamma_N,\ \forall n = 0,1,\ldots,\ N \\ \qquad \text{(flux boundary condition)} \end{cases}$$

(4.34) $$\begin{cases} P_K^0 = P_{0,K} \\ \forall\ K \in \mathcal{T}_h \\ \qquad \text{(initial condition))} \end{cases}$$

We devote the two next paragraphs 4.4 and 4.5 to the resolution of this system of equations: depending on the choice made for the main unknowns and for the unknowns which will be eliminated, we shall obtain <u>either the mixed approximation of (4.2) through (4.5) (one single flux unknown</u> Q_A <u>per edge</u>, <u>one pressure unknown</u> P_K <u>per element</u>, elimination of the pressure unknowns $TP_{K,A}$ on the edges) or its mixed hybrid approximation (one single pressure unknown TP_A per edge, elimination of the flux unknown $Q_{K,A}$ on the edges and of the pressure unknowns P_K in the elements)

For water flow problems, the mixed formulation was used first (see [4] [5] [9]), but was known to lead to a coupled linear system in two unknowns Q and P which is relatively difficult to solve as it was not positive definite (see § 4.4). The mixed-hybrid formulation was introduced later (see [6] [7]) in order to overcome this difficulty, as it leads to a simple, symmetric, positive definite linear system in the unknowns TP only (see § 4.5).

We shall investigate in paragraph 5 what happens when a quadrature rule is used to calculate the integrals appearing in the definition of the elementary matrix A_K : the set of equations (4.28) through (4.34) will reduce to the usual 5-point finite difference scheme for the approximation of the Laplacian, but with an interesting plus, namely the possibility of associating to the usual finite difference solution for pressure P a "good" approximation of the velocity field $\vec{q}$.

4.4. The mixed finite element formulation

We choose here as main unknowns for the solution of equations (4.28) through (4.34):

$$(4.35) \qquad Q_A^n, \quad A \in \mathcal{A}_h \qquad \text{(one flux value per edge)}$$

$$(4.36) \qquad P_K^n, \quad K \in \mathcal{T}_h \qquad \text{(one pressure value per element)}$$

which of course requires that we arbitrarily choose one unit normal vector on each edge A of $\mathcal{A}_h$:

$$(4.37) \qquad \vec{v}_A = \text{unit normal vector on A}, \quad A \in \mathcal{A}_h,$$

so that, for any element K of $\mathcal{T}_h$ and any edge A of $\mathcal{A}_h$ we can define.

$$(4.38) \qquad \varepsilon_{K,A} = \begin{cases} +1 & \text{if } A \subset \partial K \text{ and } \vec{v}_A \cdot \vec{v}_{K,A} = 1 \\ -1 & \text{if } A \subset \partial K \text{ and } \vec{\vartheta}_A \cdot \vec{\vartheta}_{K,A} = -1 \\ 0 & \text{if } A \not\subset \partial K. \end{cases}$$

Of course, equation (4.31) (continuity of fluxes) will be automatically satisfied as soon as

$$(4.39) \qquad Q^n_{K,A} = \varepsilon_{K,A}\ Q^n_A \quad \forall K \in \mathcal{T}_h \quad \forall A \in \mathcal{A}_h.$$

Then from boundary condition (4.33) we see that the values of Q^n_A on all edges A of Γ_N are known at every time step. For simplicity of explanation, we shall suppose throughout paragraph 4.4 that these imposed flux values are zeros :

$$(4.40) \qquad Q^n_A = 0 \quad \forall\ A \in A_h, \quad A \subset \Gamma_N$$

Hence, if we denote by :

$$(4.41) \qquad A^M_h \ = \{A \in \mathcal{A}_h \mid A \text{ is interior or } \mathcal{A} \subset \Gamma_D\}$$

(where the superscript ^{M} stands for "mixed") then we are left with the set of equations (4.28) (4.29) (4.30) (4.32) for the determination of P^n_K $K \in \mathcal{T}_h$ and Q^n_A, $A \in \mathcal{A}^M_h$ at each time step. In order to eleminate from these equations the auxiliary unknowns $TP_{K,A}$, we first express them, using the consistency equation (4.28), as functions of P_K and Q_A :

$$(4.42) \qquad \begin{cases} \forall\ K \in \mathcal{T}_h,\ \forall\ A \subset \partial K \\ TP^n_{K,A} = P^n_K - a^{-1}_K \displaystyle\sum_{A' \subset \partial K} A_{KA,A'}\ Q^n_{K,A'}. \end{cases}$$

Hence we have

$$(4.43) \qquad \begin{cases} \forall\ A \in A^M_h,\ \forall\ K \in \mathcal{T}_h, A \subset \partial K \\ \varepsilon_{K,A}\ TP^n_{K,A} = \varepsilon_{K,A}\, P^n_K - a^{-1}_K \displaystyle\sum_{A' \subset \partial K} A^M_{K,\, A,\, A'}\, Q^n_{A'} \end{cases}$$

where the 4 x 4 matrices A^M_K are defined by

$$(4.44) \qquad \begin{cases} A^M_{K,A,\, A'} \ = \varepsilon_{K,A}\ A_{K,\, A,\, A'}\, \varepsilon_{K,A'} \\ \qquad \forall\ A,\ A' \subset \partial K \end{cases}$$

(see figure 4.3, and compare to (2.9) and figure 2.1).

$$A_K^M = \frac{1}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

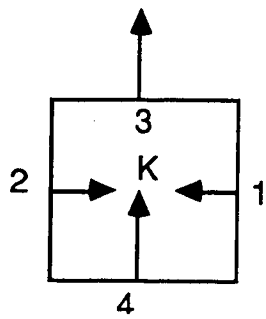


Figure 4.3. : The A_K^M matrix for the edge orientation indicated at right.

$$B^M = \frac{1}{6}\begin{array}{c} \left[\begin{array}{ccccccc} \frac{2}{a_K} & & 0 & 0 & & -\frac{1}{a_K} & \\ & & & 0 & & & \\ 0 & & & -\frac{1}{a_K} & & & \\ 0 & 0 & -\frac{1}{a_K} & \frac{2}{a_K}+\frac{2}{a_{K'}} & -\frac{1}{a_{K'}} & 0 & 0 \\ & & & -\frac{1}{a_{K'}} & & & \\ -\frac{1}{a_K} & & & 0 & & & \\ & & & 0 & & & \end{array}\right] \\ \begin{array}{ccccccc} D & F & C & A & B & E & G \end{array} \end{array} \begin{array}{c} D \\ F \\ C \\ A \\ B \\ E \\ G \\ \\ \end{array}$$

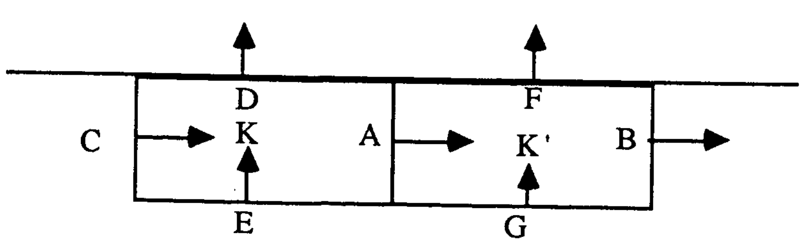


Figure 4.4 : The A^{th} and D^{th} rows and columns of the mixed matrix B^M for an interior edge A and a given pressure boundary edge D, for the edge orientation chosen as below.

We plug equation (4.43) successively into equation (4.30) (continuity of pressure on interior edges, which obviously may be rewritten $\varepsilon_{K,A}\, TP^n_{K,A} + \varepsilon_{K',A}\, TP^n_{K',A} = 0$):

$$(4.45)\qquad \begin{cases} a_K^{-1} \displaystyle\sum_{A' \subset \partial K} A^M_{K\,A,A'}\, Q^n_{A'} + a_{K'}^{-1} \displaystyle\sum_{A' \subset \partial K'} A^M_{K'A,A'}\, Q^n_{A'} = \varepsilon_{K,A} P^n_K + \varepsilon_{K',A} P^n_{K'} \\ \forall\, A \in \mathcal{A}_h,\ A \text{ interior},\ \forall n = 0,1,\dots,N \end{cases}$$

and in equation (4.32) (pressure boundary condition on Γ_D)

$$(4.46)\qquad \begin{cases} a_K^{-1} \displaystyle\sum_{A' \subset \partial K} A^M_{K\,A,\,A'}\, Q^n_{A'} = \varepsilon_{K,A}\ P^n_K - \varepsilon_{K,A} P^n_{e,A} \\ \forall\, A \in \mathcal{A}_h,\ A \subset \Gamma_D,\ \forall\, n = 0,1,\dots,N \end{cases}$$

So, if we define a <u>square matrix</u> B^M with as as many rows and columns edges A in $\mathcal{A}_h^M$ by (see figure 4.4)

$$(4.47)\qquad B^M_{A,\,A'} = \sum_{K \supset A \text{ and } A'} a_K^{-1}\ A^M_{K,A,A'}\quad \forall A,A' \in \mathcal{A}_h^M$$

(with the convention that $B^M_{A,A'} = 0$ if there exist no element K containing both A and A') and a <u>matrix DIV^M</u> with as many rows as elements K in $\mathcal{T}_h$ and as many columns as edges A in $\mathcal{A}_h^M$ by :

$$(4.48)\qquad DIV^M_{K,A} = \varepsilon_{K,A}\qquad \forall K \in \mathcal{T}_h,\ \forall\, A \in \mathcal{A}_h^M,$$

then equations (4.45) and (4.46) can be written in matrix form :

$$(4.50)\qquad B^M\ Q^n - (DIV^M)^T\, p^n + G^{M,n} = 0 \qquad n = 0,1\dots N,$$

where the superscript ^{T} stands for "transposed", and where :

$$(4.51)\qquad \begin{cases} Q^n = \text{column vector of } Q^n_A,\ A \in \mathcal{A}_h^M \\ P^n = \text{column vector of } P^n_K,\ K \in \mathcal{T}_h \\ G^{M,n} = \text{column vector of } G^{M,n}_A, A \in \mathcal{A}_h^M \end{cases}$$

with

$$(4.52)\qquad G^{M,n}_A = \begin{cases} 0 & \text{if } A \in \mathcal{A}_h,\ A \text{ interior} \\ \varepsilon_{K,A}\ P^n_{e,\,A} & \text{if } A \subset \Gamma_D,\ K \supset A. \end{cases}$$

Equation (4.50) is the sought equation resulting from the elimination of the $TP_{K,A}$ unknowns from equations (4.28) (4.30) (4.32). It has to be completed by the balance equation (4.29) on the element K, which may be rewritten in matrix form using the notation of (4.48) :

$$(4.53) \qquad M^M \frac{P^n - P^{n-1}}{\Delta t} + DIV^M Q^{n(\theta)} = F^{M,n(\theta)} \quad n = 1,2 \ldots N,$$

where the (diagonal) mixed mass matrix M^M and the right-hand side $F^{M,n\,(\theta)}$ are given by :

$$(4.54) \qquad M^M_{K,K} = |K|\, c_K \qquad \forall K \in \mathcal{T}_h$$

$$(4.55) \qquad F_K^{M,\,n\,(\theta)} = F_K^{n\,(\theta)} \qquad \forall K \in \mathcal{T}_h.$$

Summing up, we obtain for the mixed unknowns (P_K, $K \in \mathcal{T}_h$) and (Q_A, $A \in \mathcal{A}_h^M$) the

(Mixed formulation for equations (4.2. thru 5)

$$(4.50) \qquad B^M\, Q^n - (DIV^M)^T P^n + G^{M,n} = 0 \qquad n = 0,1 \ldots N$$

$$(4.53) \qquad M^M \frac{P^n - P^{n-1}}{\Delta t} + DIV^M\, Q^{n(\theta)} - F^{M,n(\theta)} = 0 \quad n=1,2\ldots N$$

$$(4.34.) \qquad P^0 = P_0$$

We investigate now the resolution of this set of equations according to the nature of the partial differential equation to be solved.

4.4.1. Mixed formulation for elliptic problems

We make here the assumption that the storage coefficient c(x) satisfies (4.11), so that

$$(4.56) \qquad c_K = 0 \quad \forall K \in \mathcal{T}_h, \text{ hence } M^M \equiv 0,$$

and we are solving a sequence of independent elliptic problems (4.2) through (4.4) which are approximated by, choosing $\theta = 1$:

$$(4.57) \qquad B^M\, Q^n - \left(DIV^M\right)^T P^n = -G^{M,n} \qquad n = 1,2 \ldots N$$

$$(4.58) \qquad - DIV^M\, Q^n = -F^{M,n} \qquad n = 1,2,\ldots,N$$

i.e. dropping the time index n :

$$(4.59)\quad \begin{bmatrix} B^M & -(DIV^M)^T \\ -DIV^M & 0 \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} -G^M \\ -F^M \end{bmatrix}$$

At each time step, we have to solve a system of the form (4.59) in the Q and P unknowns. Though it can be proved that this system has a unique solution, its matrix is symmetric but not positive definite, hence its resolution cannot be achieved by the most simple linear algebra algorithm, like the Choleski or the conjugate gradient methods. One has to resort to iterative procedures in P and Q, like the Agrumented Lagrangian Method, which can be made to work but are quite time consuming. Moreover, the number of unknowns is quite large, as both the pressure on each element and the flux through each edge have to be determined simultaneously!

4.4.2 Mixed formulation for parabolic problems

We consider here the case where the storage coefficient c(x) stays away from zero uniformly on Ω as in (4.10), so that :

$$(4.60)\qquad c_K > 0 \qquad\qquad \forall\, K \in \text{ of } \mathcal{C}_h$$

hence the mixed mass matrix M^M is diagonal and invertible. At each time step, we have to solve (compare with (4.59) :

$$(4.61)\quad \begin{bmatrix} B^M & -(DIV^M)^T \\ -\theta\, DIV^M & -\dfrac{M^M}{\Delta t} \end{bmatrix} \begin{bmatrix} Q^n \\ P^n \end{bmatrix} = \begin{bmatrix} -G^{M,n} \\ -F^{M,n}(\theta) - \dfrac{M^M}{\Delta t} P^{n-1} + (1-\theta)\, DIV^M\, Q^{n-1} \end{bmatrix}$$

The resolution of system (4.61) will differ according to the value of θ chosen for the time discretization :

• Explicit time discretization ($\theta = 0$)

(which in the usual finite difference scheme leads to an explicit calculation of time step n from time step n -1) Then the system (4.61) in P^n and Q^n decouples : one first calculates P^n by the explicit (as M^M is diagonal) formula:

$$(4.62)\qquad M^M P^n = M^M\, P^{n-1} - \Delta t\, DIV^M\, Q^{n-1} + \Delta t\, F^{M,n-1}$$

and then Q^n is given by the resolution of the symmetric positive definite system :

$$(4.63) \quad B^M\, Q^n = \left(DIV^M\right)^T P^n - G^{M,n}$$

The situation is somewhat better than in the elliptic case, as any standard method can be used for the resolution of the linear system (4.63), whose number of unknowns (one flux value Q_A per edge of $\mathcal{A}_h^M$) is moreover much smaller. Compared now to the classical finite difference method, the mixed approach (4.62) (4.63) has lost the simplicity -and efficency- of the explicit calculations. But this has to be tempered by the fact that , for a rectangular mesh as we consider here, the matrix B^M has a very nice structure, as seen in figure 4.4 : the linear system (4.63) can be split into $N_1\, N_2$ independent symmetric positive definite subsystems associated to each row and column of edges of the mesh (see figure 3.1). These subsystems can be made tridiagonal by an adequate numbering of the edges (first by rows, then by columns for example), so that the Choleski decomposition of each of these subsystems can be easily stored, reducing the resolution of (4.63) to a sequence of $N_1\ N_2$ independant up-down calculations. The independence of these $N_1\ N_2$ subsystems makes it also possible to take advantage of the parallelism ability of the computer when available.

• Implicit time discretization ($\theta > 0$)
(as for example $\theta = .5$ in the popular Crank-Nicholson scheme). In this case, we can use the second equation of (4.61) to calculate P^n in term of Q^n, P^{n-1}, Q^{n-1}, and plug it into the first equation of (4.61) :

$$(4.63\text{ bis}) \quad \begin{aligned} &\left[B^M + \theta\Delta t \left(DIV^M\right)^T\left(M^M\right)^{-1} DIV^M\right] Q^n = \\ &\left(DIV^M\right)^T\ P^{n-1} - (1-\theta)\, \Delta t \left(DIV^M\right)^T \left(M^M\right)^{-1}\ DIV^M\, Q^{n-1} \\ &+\Delta t\, DIV^T \left(M^M\right)^{-1}\ G^{M,n\,(\theta)} - G^{M,n} \end{aligned}$$

Once again, we are led to solve at each time step one system similar to (4.63) in the unknowns Q_A^n, $A \in \mathcal{A}_h^M$, but with a much less coarse matrix, which does not decouple into independent subsystems as in (4.63)!

Moreover when the storage coefficient c becomes small, the conditionning of the linear system (4.63 bis) become worse and worse, because the matrix $(M^M)^{-1}$ blows up to infinity. .

Summarizing the situation for the parabolic case, we see that using a mixed formulation leads always; whatever value of θ is used for the time discretization, to the resolution at each time step of a symmetric positive definite system for the flux unknowns $\left(Q_A,\ A \in \mathcal{A}_h^M\right)$:

when $\theta = 0$, this system decouples into N_1 N_2 independent subsystems associated to rows and columns of edges.

when $\theta > 0$ this system has to be solved as a whole, and becomes singular when the storage coefficients c_K go to zero.

4.5 The mixed-hybrid finite element formulation

We choose here as main unknowns for the solution of equations (4.28) through (4.34).

$$(4.64) \qquad TP_A, \quad A \in \mathcal{A}_h$$

(one pressure value per edge)

Of course, equation (4.30) (continuity of pressure) will be automatically satisfied as soon as :

$$(4.65) \qquad TP_{K,A} = TP_A \quad \forall A \in \mathcal{A}_h, \forall K \supset A$$

and from boundary condition (4.32) we see that the values of TP_A^n are known on all edges A of Γ_D. For sake of simplicity, we shall suppose throughout paragraph 4.5 that

$$(4.66) \qquad TP_A^n = 0 \qquad \forall A \in \mathcal{A}_h, \ A \subset \Gamma_D$$

Hence, if we denote by :

$$(4.67) \qquad \mathcal{A}_h^{MH} = \{A \in \mathcal{A}_h \mid A \text{ is interior or } A \subset \Gamma_N\}$$

(where the superscript ^{MH} stands for "mixed-hybrid") then we are left with the set of equations (4.28) (4.29) (4.31) (4.33) for the determination of TP_A^n, $A \in \mathcal{A}_h^{MH}$ at each time step. In order to eliminate from there equations the auxiliary unknowns P_K and $Q_{K,A}$, we first express these latter, using the consistency equation (4.28), as functions of P_K and TP_A :

$$(4.68) \qquad \begin{cases} \forall K \in \mathcal{T}_h, \ \forall A \subset \partial K \\ Q_{K,A}^n = a_K \left(\alpha_{K,A} P_K^n - \sum\limits_{A' \subset \partial K} A_{K,A,A'}^{-1} \, TP_{A'}^n \right) \end{cases}$$

where the numbers $\alpha_{K,A}$ represent the sum of the elements of the A-th row of the A_K^{-1} matrix :

$$(4.68\text{-}1)\quad \begin{cases} \forall K \in \mathcal{T}_h, \quad \forall A \subset \partial K \\ \alpha_{K,A} = \sum\limits_{A' \subset \partial K,A} A^{-1}_{KA,A'} \end{cases}$$

(and also the sum of the elements of columns as A_K^{-1} is symmetric). We shall use also a number α_K representing the sum of all elements of A_K^{-1} :

$$(4.68\text{-}2)\qquad \forall K \in \mathcal{T}_h \quad \alpha_K = \sum_{A \subset \partial K} \alpha_{K,A}$$

<u>The numbers $\alpha_{K,A}$ and α_K are dimensionless shape factors associated to each element K</u>

When the elements are square , as we have supposed here, we see from (2.10) that :

$$(4.68\text{-}3)\quad \begin{cases} \alpha_{K,A} = 2 \quad \forall K \in \mathcal{T}_h \ \forall A \subset \partial K \\ \alpha_K \ = 8 \quad \forall K \in \mathcal{T}_h \end{cases}$$

However we shall keep in the sequel the notations $\alpha_{K,A}$ .and α_K in order to get formula valid when the elements K are rectangular and not square.

We plug now the expression (4.68) of $Q^n_{K,A}$ into the three remaing equations (4.28) through (4.34) containing the unknowns $Q^n_{K,A}$:

i) <u>in the balance equation</u> (4.29) :
We notice first using (4.68), (4.68-1) and the fact that the matrix A_K^{-1} is symmetric :

$$(4.69)\qquad \mathrm{DIV}_K\ Q^n_K = \sum_{A \subset \partial K} Q^n_{K,A} = a_K \left(\alpha_K P^n_K - \sum_{A \subset \partial K} \alpha_{K,A}\, TP^n_A \right)$$

which we plug into (4.29):

$$(4.70)\quad \begin{cases} |K|\ c_K \dfrac{P^n_K - P^{n-1}_K}{\Delta t} + a_K \left(\alpha_K P^{n(\theta)}_K - \sum\limits_{A \subset \partial K} \alpha_{K,A}\, TP^{n(\theta)}_A \right) = F^{n(\theta)}_K \\ \forall K \in \mathcal{T}_h, \qquad \forall n = 1,2,\ldots,N \end{cases}$$

ii) <u>in equation (4.31) for the continuity of fluxes :</u>

$$
(4.71)\quad \begin{cases} \alpha_{K,A}\, a_K\, P_K^n + \alpha_{K',A}\, a_{K'}\, P_{K'}^n = a_K \sum\limits_{A'\subset\partial K} A^{-1}_{K,A,A'}\, TP^n_{A'} \\ \qquad\qquad + a_{K'} \sum\limits_{A'\subset\partial K'} A^{-1}_{K',A',A'}\, TP^n_{A'} \\ \forall\ A\in \mathcal{A}_h,\quad A \text{ interior} \end{cases}
$$

iii) in the flux boundary condition (4.33)

$$
(4.72)\quad \begin{cases} \alpha_{K,A}\ a_K\, P_K^n = a_K \sum\limits_{A'\subset\partial K} A^{-1}_{K,A,A'}\, TP^n_{A'} + Q^n_{e,A} \\ \forall\ A\in \mathcal{A}_h,\quad A\subset\Gamma_N. \end{cases}
$$

We have now to eliminate the element pressures P_K^n from equations (4.70) through (4.72) in order to obtain equations for the edge pressures TP_K^n only. For sake of clarity, we distinguish now the elliptic and parabolic cases.

4.5.1. Mixed-hybrid finite element formulation for elliptic problems

In such problems we have $c_K = 0$. Then choosing $\theta = 1$, the equation (4.70) reduces, dropping the time index n, to :

$$
(4.73)\quad a_K\left(\alpha_K\, P_K - \sum_{A'\subset\partial K} \alpha_{K,A'}\ TP_{A'}\right) = F_K \qquad\qquad \forall\ K\in \mathcal{T}_h,
$$

As we need in (4.71) (4.72) the quantities $\alpha_{K,A}\, P_K$ instead of the quantities $\alpha_K\, P_K$ which comes naturally out of (4.73), we define :

$$
(4.73\text{-}1)\quad \begin{cases} v_{K,A,A'} = \dfrac{\alpha_{K,A}\, \alpha_{K,A'}}{\alpha_K} \qquad \left(\equiv \dfrac{1}{2} \text{ for square K}\right) \\ \forall\ K\in \mathcal{T}_h, \qquad \forall\ A, A'\subset\partial K \end{cases}
$$

The elimination of the element pressure P_K is then straightforward : dividing (4.73) by α_K multiplying it by $\alpha_{K,A}$ and plugging it into (4.71) (4.72) yields :

$$(4.74)\quad \begin{cases} a_K \sum\limits_{A' \subset \partial K} v_{K,A,A'}\, TP_{A'} + a_{K'} \sum\limits_{A' \subset \partial K'} v_{K'A,A'}\, TP_{A'} \\ \qquad + \dfrac{\alpha_{K,A}}{\alpha_K} F_K + \dfrac{\alpha_{K',A}}{\alpha_{K'}} F_{K'} \\ \qquad\qquad = a_K \sum\limits_{A' \subset \partial K} A^{-1}_{K,A,A'}\, TP_{A'} + a_{K'} \sum\limits_{A' \subset \partial K'} A^{-1}_{K,A,A'}\, TP_{A'} \\ \forall\, A \in \mathcal{A}_h,\ A \text{ internal} \end{cases}$$

$$(4.75)\quad \begin{cases} a_K \sum\limits_{A' \subset \partial K} v_{K,A,A'}\, TP_{A'} + \dfrac{\alpha_{K,A}}{\alpha k} F_K = a_K \sum\limits_{A' \subset \partial K} A^{-1}_{K,A,A'}\, TP_{A'} + Q^n_{e,A} \\ \forall\, A \in \mathcal{A}_h,\quad A \subset \Gamma_N \end{cases}$$

which is the sought system of equation for the edge pressures unknowns TP_A, $A \in \mathcal{A}^{MH}_h$. In order to write it in a matrix form, we define :

$$(4.76)\quad \begin{cases} M^{MH} = \text{mixed-hybrid mass matrix} \\ M^{MH}_{A,A'} = \sum\limits_{K \supset A \text{ and } A'} a_K\, A^{-1}_{K,A,A'} \quad \forall\, A,A' \in \mathcal{A}^{MH}_h \end{cases}$$

and

$$(4.77)\quad \begin{cases} N^{MH} = \text{mixed-hybrid rigidity matrix} \\ N^{MH}_{A,A'} = \sum\limits_{K \supset A \text{ and } A'} a_K\, v_{K,A,A'} \quad \forall\, A,A' \in \mathcal{A}^{MH}_h \end{cases}$$

$$(4.78)\quad F^{MH}_A = \sum_{K \supset A} \frac{\alpha_{K,A}}{\alpha_K} F_K \qquad \forall\, A \in A^{MH}_h$$

$$(4.79)\quad G^{MH}_A = \begin{cases} 0 & \text{if } A \text{ interior} \\ Q_{e,A} & \text{if } A \subset \Gamma_N \end{cases} \qquad \forall\, A \in \mathcal{A}^{MH}_h$$

so that equation (4.74) (4.75) may be rewritten in matrix form :

Mixed-Hybrid Formulation for the elliptic problem

$$(4.80)\quad (M^{MH} - N^{MH})\, TP = F^{MH} - G^{MH}$$

The matrices M^{MH} and N^{MH} are illustrated in figures 4.5 and 4.6. They are obviously symmetric, and $M^{MH} - N^{MH}$ can be proved to be positive definite, allowing thus for an easy resolution of the linear equation (4.80)

$$
M^{MH} = 2
\begin{array}{cc}
\left[\begin{array}{ccccccc}
2a_K & & 0 & 0 & & & -a_K \\
 & & & 0 & & & \\
0 & & & -a_K & & & \\
0 & 0 & -a_K & 2a_K + 2a_{K'} & -a_K & 0 & 0 \\
 & & & -a_K & & & \\
-a_K & & & 0 & & & \\
 & & & 0 & & &
\end{array}\right]
&
\begin{array}{c}
D \\ F \\ C \\ A \\ B \\ E \\ G
\end{array}
\\
\begin{array}{ccccccc}
D & F & C & A & B & E & G
\end{array}
&
\end{array}
$$

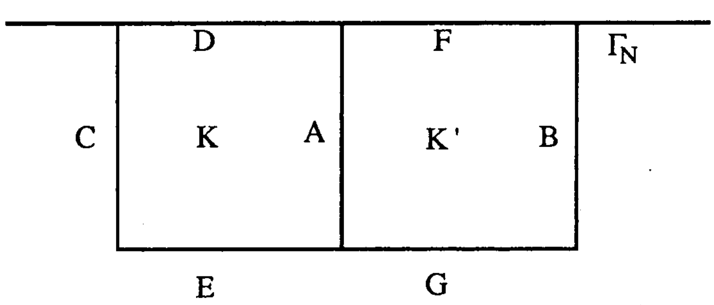


Figure 4.5 : The A^{th} and D^{th} rows and columns of the mixed-hybrid mass matrix M^{MH} for an interior edge A and a given flux boundary edge D, in the case of square elements K and K'.

$$
N^{MH} = \frac{1}{2}
\begin{array}{c}
\left[\begin{array}{ccccccc}
a_K & & a_K & a_K & & a_K & \\
 & & & & & & \\
a_K & & & & & & \\
a_K & a_{K'} & a_K & a_K + a_{K'} & a_{K'} & a_K & a_{K'} \\
 & & & & & & \\
a_K & & & & & & \\
 & & & & & &
\end{array}\right]
\\
\begin{array}{ccccccc}
D & F & C & A & B & E & G
\end{array}
\end{array}
\begin{array}{c}
D \\ F \\ C \\ A \\ B \\ E \\ G \\
\end{array}
$$

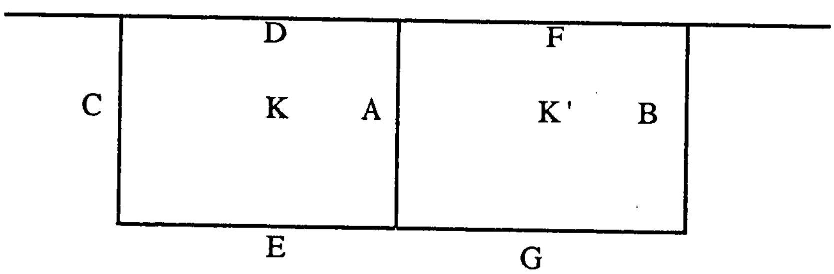


Figure 4.6 : The A^{th} and D^{th} rows and columns of the mixed-hybrid rigidity matrix N^{MH} for an interior edge A and a given flux boundary edge D, in the case of square elements K and K'.

Comparing the mixed-hybrid equation (4.80) to the corresponding mixed equations (4.59), we see the great improvement brought by the mixed-hybrid approach upon the mixed approach for the resolution of stationary elliptic problems : the level of difficulty for the resolution of the mixed equation (4.80) is similar to that for classical finite differences the main differences being the larger number of unknowns (roughly twice as many edges as elements) and the less sparse matrix (seven star scheme instead of five star).

Once the edge pressures TP have been computed by solving (4.80), the pressures P_K and the fluxes $Q_{K,A}$ are computed easily by local calculations on each element K using (4.73) and (4.68). If one is interested only in the velocity field $\vec{q}_h$, it will be enough to make these local calculations only on the black elements of a black and white coloring of the mesh $\mathcal{T}_h$.

4.5.2 Mixed-hybrid formulation for parabolic problems

We consider now the full equation (4.70) with a storage coefficient $c_K > 0$ on all elements K. In order to eliminate again P_K^n from (4.70 thru 72), we rewrite first (4.70) in a more suitable way. We define :

$$(4.81) \quad \begin{cases} \forall K \in \mathcal{T}_h : \quad \lambda_K = \alpha_K \dfrac{a_K \Delta t}{c_K |K|} \quad \text{(dimensionless)} \\ \qquad\qquad \beta_K = \dfrac{\lambda_K}{1 + \theta \lambda_K} \\ \forall A, A' \subset \partial K : \nu_{K,A,A'} = \beta_K \dfrac{\alpha_{K,A}\, \alpha_{K,A'}}{\alpha_K} \end{cases}$$

Hence multiplying (4.70) by $\dfrac{\beta_K}{\alpha_K a_K}$ yields

$$(4.82) \qquad P_K^n = (1-\beta_K)\, P_K^{n-1} + \beta_K \sum_{A' \subset \partial K} \frac{\alpha_{K,A'}}{\alpha_K}\, TP_{A'}^{n(\theta)} + \frac{\beta_K}{\alpha_K a_K} F_{K\,A}^{n(\theta)}$$

Multiplying (4.82) by $\alpha_{K,A}\, a_K$ we obtain :

$$(4.83) \qquad \alpha_{K,A}\, a_K\, P_K^n = \alpha_{K,A}\, a_K\, (1-\beta_K)\, P_K^{n-1} + a_K \sum_{A' \subset \partial K} \nu_{K,A'A}\, TP_{A'}^{n(\theta)} + \beta_K \frac{\alpha_{K,A}}{\alpha_K} F_K^{n(\theta)}$$

Plugging then (4.83) into (4.71) (4.72) we obtain the sought equations for the sole edge pressure unknowns TP_A^n, $A \subset A_h^{MH}$

$$(4.84)\quad \sum_{K\supset A} \alpha_{K,A}\, a_K \left(1-\beta_K\right) P_K^{n-1}$$

$$+ \sum_{K\supset A} a_K \sum_{A'\subset\partial K} \nu_{K,A,A'}\, TP_{A'}^{n\,(\theta)}$$

$$+ \sum_{K\supset A} \beta_K \frac{\alpha_{K,A}}{\alpha_K} F_K^{n(\theta)}$$

$$= \sum_{K\supset A} a_K \sum_{A'\subset\partial K} A^{-1}_{K,A,A'}\, TP^{n}_{A'}$$

$$+ \begin{cases} 0 \text{ if A internal} \\ Q^{n}_{e,A} \text{ if } A\in\Gamma_N \end{cases}$$

$$\forall A \in \mathcal{A}_h^M$$

In order to write (4.87) in a matrix form, we define :

(4.85) M^{MH} = mixed-hybrid rigidity mass matrix as in (4.76)

(4.86) $N^{MH} = \begin{cases} \text{mixed-hybrid rigidity matrix as in (4.77)} \\ \text{but of course with } a_{K;\,A,A'} \text{ defined by (4.81)} \end{cases}$

$$(4.87)\quad F_A^{MH,n\,(\theta)} = \sum_{K\supset A} \left\{\beta_K \frac{\alpha_{K,A}}{\alpha_K} F_K^{n(\theta)} + \alpha_{K,A}\left(1-\beta_K\right) P_K^{n-1}\right\} \quad \forall A\in A_h^{MH}$$

$$(4.88)\quad G_A^{M\,H,n} = \begin{cases} 0 & \text{if A is internal} \\ Q^{n}_{e,A} & \text{if } A\in\Gamma_N \end{cases} \qquad \forall A \in \mathcal{A}_h^M$$

Comparing with the definitions (4.76 through 4.79) used in the elliptic case, one remarks that when the storage coefficient $c_K \to 0$ and $\theta = 1$, then $\lambda_K \to +\infty$ and $\beta_K \to 1$ so that N^{MH} and $F_A^{MH,n}$ defined by (4.86) (4.87) converge towards their corresponding elliptic definitions (4.77) (4.78)

Now we can rewrite equation (4.84) as :

Mixed-hybrid Formulation for the parabolic problems

$$(4.89)\quad \left(M^{MH} - \theta\, N^{MH}\right) TP^{n} = (1-\theta)\, N^{MH}\, TP^{n-1} + F^{MH,n(\theta)} - G^{MH,n}$$

The matrix M^{MH} has already been illustrated in figure 4.5, and the matrix N^{MH} is similar to that of figure 4.6, only replacing everywhere a_K by $\beta_K a_K$. Once again, the matrix $M^{MH} - \theta N^{MH}$ can be proven to be (symmetric)positive definite, so that any standard method can be used for the resolution of (4.89) Once the edge pressure TP^n_A, $A \in \mathcal{A}_h^{MH}$ have be computed by solving (4.89), the element pressures P^n_K, $K \in \mathcal{T}_h$ have to be calculated locally on each element K using the explicit formula (4.82), and the whole procedure can be started again for the next time step. If the velocity field q_h is needed, its degrees of freedom $Q^n_{K,A}$ are immediately obtained from the explicit formula (4.68).

We discuss now the advantages of the mixed-hybrid approximation (4.89) over the mixed approximation (4.61) for parabolic problems :

• Explicit time discretization ($\theta = 0$).

Then (4.89) reduces to

$$(4.90) \qquad M^{MH}\ TP^n = N^{MH}\ TP^{n-1} + F^{MH,n-1} - G^{MH,n}$$

which is not explicit, as it would be in the case of the usual finite difference scheme, but is not very far from that, as we can see in figures 4.5 and 4.4 that M^{MH} and B^M have the same structure. So the resolution of (4.90) reduces to the solution of $N_1 N_2$ independent subsystems associated to rows and columns of edges of the mesh, exactly as for the mixed approximation (4.63), to which we refer for the resolution of (4.89).

• Implicit time discretization ($\theta > 0$).

Once again, the matrix $M^{MH} - \theta N^{MH}$ can be proved to be (symmetric) positive definite, so that the solution of (4.89) can be achieved by any method.

Comparing the mixed-hybrid approximation (4.89) and the mixed approximation (4.63 bis), we see that both systems have roughly the same number of unknowns, and that their matrices have the same structure, so the two approaches seem comparable. But the main advantage of the mixed-hybrid formulation (4.89) over the mixed formulation (4.63 bis) appears when the storage coefficients c_K tend to zero (with $\theta = 1$) : in that case the mixed-hybrid equation (4.89) tends toward the elliptic mixed-hybrid equation (4.80), which is known to be well posed, whereas the mixed equation (4.63 bis) becomes singular.

Summarizing the situation for the parabolic case, we see that the mixed-hybrid formulation is very similar to the previously investigated mixed formulation : whatever value of θ is used for the time discretization, one has to solve at each time step a symmetric positive definite system for the edge pressures $\left(TP_A, A \in A_h^{MH}\right)$:

When $\theta = 0$, this system decouples into $N_1 N_2$ independant subsystems associated to rows and columns of edges.

When $\theta > 0$, this system has to be solved as a whole, and tends to the well-posed elliptic mixed-hybrid formulation when the storage coeffcients c_K go to zero.

Hence the mixed-hybrid formulation has to be prefered to the mixed formulation especially for problems which have both elliptic and parabolic regions.

5. Using a quadrature rule : back to good old finite differences, with a plus.

We investigate in this paragraph the effect, on the finite element equations of paragraph 4.3, of using on the numerical quadrature rule :

$$(5.1) \qquad \int_K \varphi(x)\, dx \approx \frac{|K|}{4} \sum_{i=1}^{4} \varphi(M_i)$$

(where the M_i are the four vertices of element K) whenever such an integral is required. Looking back at the way these equations have been established, we see that we have used integrals over K at only three places :

• in paragraph 2 when we computed the scalar products A_{ij} of the basis $\vec{w}_j(x)$ of the Raviart-Thomas space $\vec{X}_K$. One sees immediately that

$$\int_K \vec{w}_1(x).\vec{w}_1(x)\, dx \approx \frac{|K|}{4}\left(\frac{1}{h^2} + \frac{1}{h^2} + 0 + 0\right) = \frac{1}{2}$$

$$\int_K \vec{w}_1(x).\vec{w}_2(x)\, dx \approx \frac{|K|}{4}(0+0+0+0) = 0$$

so that the matrix A_K becomes diagonal :

$$(5.2) \qquad A_K = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_K^{-1} = 2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• in paragraph 3 when we established the consistency equation (3.12) to be verified by $\vec{q}_h$, P_K and TP_K. Replacing then in (3.12) the integral over K by the quadrature formula does not change the R.H.S. of (3.12) as the function $P_K \nabla \vec{s}_K$ is constant over K and hence its integral is calculated exactly by (5.1), but changes the L.H.S. of (3.12), as the function $\frac{\vec{q}_K \cdot \vec{s}_K}{a_K}$ is quadratic over K and hence integrated only approximately by (5.1).Equation (3.12) may still be written in matrix form as (3.14) but with the matrix A_K being given by (5.2) ! Hence this consistency equation (3.14) (or (4.28))reduces to :

$$(5.3) \quad \begin{cases} Q^n_{K,A} = 2a_K\left(P^n_K - TP^n_{K,A}\right) \\ \forall\, K \in \mathcal{T}_h, \quad \forall\, A \subset \partial K, \quad \forall\, n = 0,1\dots N \end{cases}$$

• in paragraph 4.2, where the balance equation on K was first defined in its variational form (4.18 bis). As we already noticed in paragraph 4.2, all functions to be integrated in (4.18 bis). are constant over K, so that their integrals are calculated exactly by the quadrature rule (5.1) hence the balance equation (4.19) (or (4.29)) remains unchanged.

In conclusion , the discrete equations (4.28) through (4.34) of paragraph 4.3 remain valid when the quadrature rule (5.1) is used, provided that the consistency equation (4.28) is replaced by (5.3).

In order to solve these equations, we may choose now as main unknowns:

$$(5.4) \quad P^n_K, \qquad K \in \mathcal{T}_h$$

and eliminate the other unknowns $TP^n_{K,A}$ and $Q^n_{K,A}$.
We consider for that (see figure 5.1) an interior edge A and a given pressure boundary edge D:

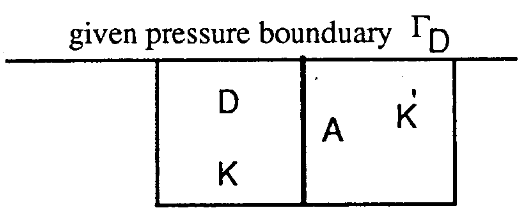


Figure 5.1 Notation for the elimination of $TP^n_{K,A}$ and $Q^n_{K,A}$.

• Using twice (5.3) on A we obtain

$$(5.5) \qquad \frac{1}{2aK} Q^n_{K,A} = P^n_K - TP^n_{K,A}$$

$$(5.6) \qquad \frac{1}{2aK'} Q^n_{K',A} = P^n_{K'} - TP^n_{K',A}$$

Taking the difference (5.5) - (5.6) yields, using the continuity of flux (4.31) and of pressure (4.32) :

$$(5.7) \qquad \left(\frac{1}{2aK} + \frac{1}{2\ aK'}\right) Q^n_{K,A} = P^n_K - P^n_{K'}.$$

<u>Defining then $a_{K,K'}$ as the harmonic mean of a_K and</u> $a_{K'}$:

$$(5.8) \qquad a^{-1}_{K,K'} = \frac{1}{2} a^{-1}_K + \frac{1}{2} a^{-1}_{K'}$$

we may rewrite (5.7) as :

$$(5.9) \qquad Q^n_{K,A} = a_{K,K'} (P^n_K - P^n_{K'})$$

which is the usual finite difference formula for the flux from K to K' :

$$Q^n_{K,A} = h\ a_{K,K'} \left(\frac{P^n_K - P^n_{K'}}{h}\right)$$

where the first h comes from the integration along the edge A.

• Using one time (5.3) on D we obtain, using the pressure boundary condition (4.32) :

$$(5.10) \qquad Q^n_{K,D} = 2a_K (P^n_K - P^n_{e,D})$$

which once again can be interpreted as a finite difference formula :

$$Q^n_{K,D} = h\ a_K \left(\frac{P^n_K - P^n_{e,D}}{h/2}\right).$$

Now we can eliminate completly the unknowns $Q^n_{K,A}$ form the balance equation (4.29) using (5.9) on interior edges, (5.10) on boundary edges of Γ_D, and the boundary condition (4.33) on boundary edges of Γ_N, which yields the :

Finite Difference Approximation of equations (4.2 through 5):

$$(5.11)\quad |K|\, c_K \frac{P_K^n - P_K^{n-1}}{\Delta t} + \sum_{K' \in N(K)} a_{K,K'}\left(P_K^{n\,(\theta)} - P_{K'}^{n\,(\theta)}\right) + \sum_{\substack{A \subset \partial K \\ A \subset \Gamma_D}} 2a_K\left(P_K^{n\,(\theta)} - P_{e,A}^{n\,(\theta)}\right) + \sum_{\substack{A \subset \partial K \\ A \subset \Gamma_N}} Q_{e,A}^{n(\theta)} = F_K^{n(\theta)}, \quad \forall K \in \mathcal{T}_h,\ \forall n = 1,2 \ldots N$$

where

$$(5.12)\quad N(K) = \{K' \in \mathcal{T}_h \mid K \cap K' = \text{one edge}\}.$$

We recognize the usual five stars Finite Difference scheme with harmonic interbloc transmissivities.

So the question may rise of the interest of developping the relatively heavy artillery of mixed and mixed hybrid finite element if the final result is only the well-known finite difference scheme ! The answers is two-fold :

• first, the appoximation technique developped in paragraph 4 for the mixed-hybrid formulation has an interest in itself : it yields a more precise approximation than the finite difference approach, and morever can be generalized in various ways to irregular meshes, higher order elements, etc...
In that sense, the mixed or mixed-hybrid F.E. method is the natural generalization of the finite difference method.

• Second, even if one is interested only in the F.D. scheme, its approach as a particular case of a mixed finite element approximation brings an information, which is completely lacking in the usual F.D. approch, on <u>how to define a consistent approximation $\vec{q}_h(x)$ of the velocity field</u> $\vec{q}(x)$ which retains the important features of $\vec{q}$ (continuity of fluxes through edges) and gives a conservative scheme : the velocity field $\vec{q}_h(x)$ <u>associated to the classical F.D. solution (P_K^n</u> K $\in \mathcal{T}_h$) <u>of (5.11) is immediately given, on one element K of $\mathcal{T}_h$ by:</u>

$$(5.13)\quad \vec{q}_h^{\,n}(x) = \sum_{A \subset \partial K} Q_{K,A}^n \vec{w}_A(x)$$

where the $\overrightarrow{w}_A(x)$ functions are given by (2.7), and where the flux values $Q_{K,A}^n$ are given by (5.9). (5.10), or (4.33) according to the nature of the edge A!

To conclude, let us mention a side-advantage of the above approach to finite differences : the bounduary condition are automatically discretized in such away that the rigidity matrix associated to the F.D. scheme (5.11) is always symmetrical positive definite (this is not necessarily the case with classical Finite Differences).

6- Generalizations

For sake of simplicity, we have considered in this paper the most elementary case where cells or elements are all squares of the same size h. But the techniques discribed above can be generalized in many ways :

• Of course, one is not bound to use square elements ! In practice, one uses rectangular elements, of size h in the x_1 direction and k in the x_2 direction. Then the whole theory goes over provided the matrix A_K in (2.9) is replaced by :

$$(6.1) \qquad A_K = \frac{1}{6}\begin{bmatrix} 2\frac{h}{k} & -\frac{h}{k} & 0 & 0 \\ -\frac{h}{k} & 2\frac{h}{k} & 0 & 0 \\ 0 & 0 & 2\frac{k}{h} & -\frac{k}{h} \\ 0 & 0 & -\frac{k}{h} & 2\frac{k}{h} \end{bmatrix}$$

• Generalization to 3-D rectangular meshes is straightforward : edges are replaced by faces, the unknowns being then the element pressures P_K, the face pressures TP_A and the flow-rate Q_A through a face A.

If the elements K are cubes of side length h, then the basis functions of the Raviart-Thomas space $\vec{X}_K$ are (compare with (2.7)

$$\vec{w}_1(x) = \begin{bmatrix} x_1/h^3 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{w}_2(x) = \begin{bmatrix} (h-x_1)/h^3 \\ 0 \\ 0 \end{bmatrix}$$

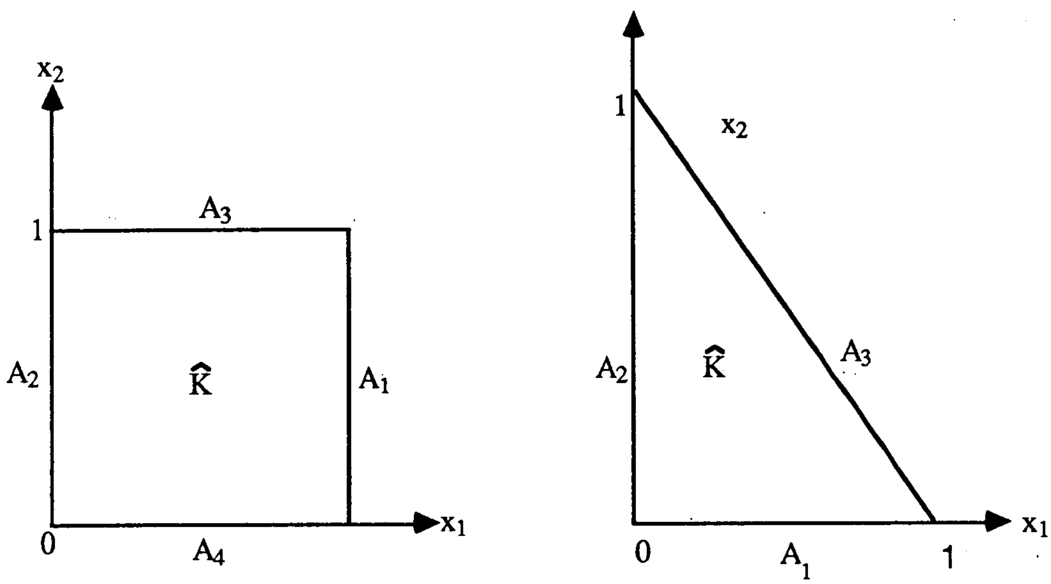
$$\vec{w}_3(x) = \begin{bmatrix} 0 \\ x_2/h^3 \\ 0 \end{bmatrix} \qquad \vec{w}_4(x) = \begin{bmatrix} 0 \\ (h-x_2)/h^3 \\ 0 \end{bmatrix}$$

$$\vec{w}_5(x) = \begin{bmatrix} 0 \\ 0 \\ x_3/h^3 \end{bmatrix} \qquad \vec{w}_6(x) = \begin{bmatrix} 0 \\ 0 \\ (h-x_3)/h^3 \end{bmatrix}$$

so that the A_K matrix (2.9) is replaced by a 6×6 matrix with three 2×2 diagonal blocks.

• Generalization to 2.-D irregular meshes is also quite easy for meshes made up of parallelograms and triangles only. The 4×4 (for parallelograms) or 3×3 (for triangles) matrix A_K are easily calculated in that case using <u>the affine transformation</u> which maps the element K into the corresponding reference element $\hat{K}$ shown in figure 6.1. The basis functions of the Raviart-Thomas space for the reference square are given by (2.7) with h = 1, the ones for the reference triangle are :

$$(6.3) \quad \vec{w_1}(x) = \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix} \quad \vec{w_2}(x) = \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \quad \vec{w_3}(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



<u>Figure 6.1.</u> : The reference elements of a 2D irregular mesh, with edge numbering.

For mnemonic purpose, notice the above basis functions can be visualized very easily (cf.figure 6.2) :

(6.4) in a parallelogram K , if we denote by M_j the projection of $M(x_1, x_2)$ onto the edge opposite to A_j in the direction parallel to the edges adjacent to A_j, then one has $\vec{w_j}(x_1,x_2) = \overrightarrow{M_jM}$

(6.5) in a triangle K, if we denote by M_j the vertex opposite the edge A_j, then one has $\vec{w_j}\ (x_1,x_2) = \overrightarrow{M_j\ M}$

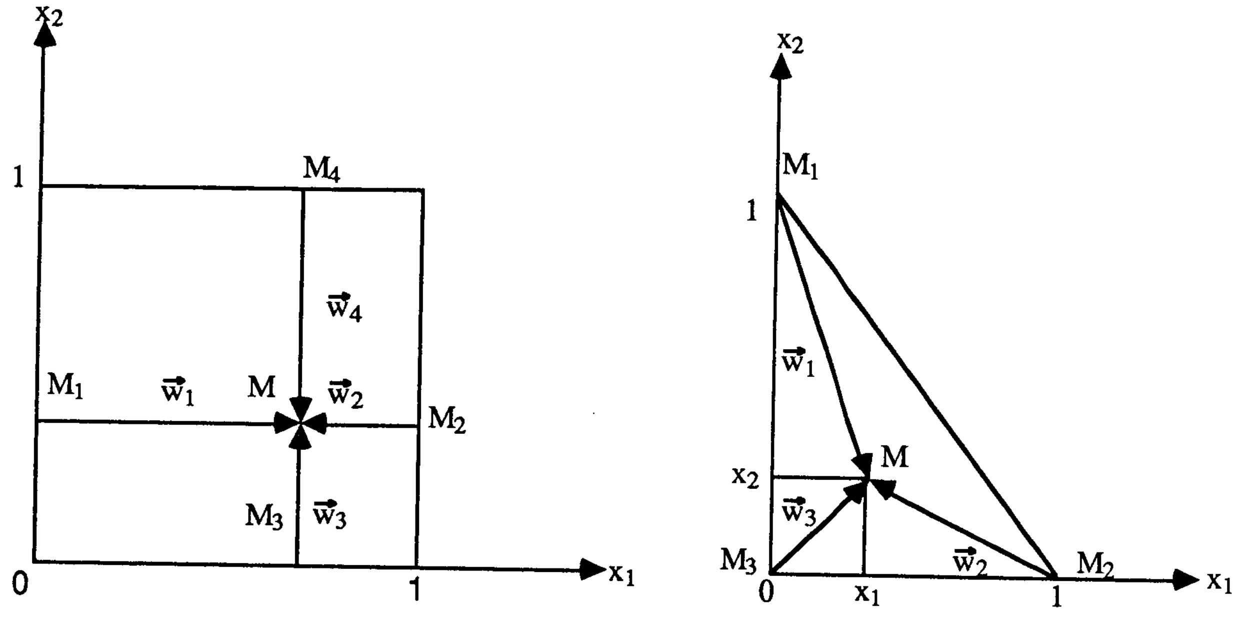


Figure 6.2 The Raviart-Thomas basis functions w_j on the reference elements.

All comments on the comparison of the mixed, mixed-hybrid and finite difference methods made in paragraph 4 carry over to irregular meshes, the only changes being of course that neither the mixed-hybrid matrix B^M of (4.47) or the mixed-hybrid mass matrix M^{MH} of (4.76) retains block-diagonal structure (one block per row and column of the rectangular mesh), which is a strict attribute of rectangular meshes. Hence, if one uses an "explicit" scheme ($\theta = 0$) for the solution of a parabolic problem on an irregular mesh, one has to solve at each time step one linear system with matrix B^M (if one uses the mixed formulation) or M^{MH} (if one uses the mixed-hybrid formulation), which deprives the $\theta = 0$ scheme of most of its attractive power, as explicit calculations are completely lost. However, as we noticed in paragraph 5, using on each element an ad-hoc (but low precision) quadrature formula yields a diagonal matrix A_K on rectangular elements, and allows to reduce the solution of the mixed or mixed-hybrid formulation to that of the classical finite difference scheme (with the advantage that an approximation $\vec{q_h}$ of the velocity field could be associated in a natural way to the finite difference solution). We can try the same idea for an irregular mesh, using the formula

$$(6.6) \qquad \int_K \varphi(x)\,dx \simeq \frac{|K|}{4} \sum_{i=1}^{4} \varphi(M_i)$$

for a parallelogram, and

$$(6.7) \qquad \int_K \varphi(x)\,dx \simeq \frac{|K|}{3} \sum_{i=1}^{3} \varphi(M_i)$$

for a triangle (where $|K|$ is the area and M_i $i = 1,\ldots,3$ the vertices of the element). Then the matrices A_K become, on the reference elements $\hat{K}$:

$$(6.8) \qquad A_{\hat{K}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ on the reference square(as in (5.2))}$$

$$(6.9) \qquad A_{\hat{K}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \text{ on the reference triangle}$$

and we see that the matrix $A_{\hat{K}}$ is no longer diagonal on triangles ! Hence using the low-order quadrature formula (6.6) (6.7) does not ead to simplifications on an irregular mesh : in that case, exact integration - or in practice higher order quadrature formulas such as the gauss formula - have to be preferred.

In conclusion, when going to irregular meshes, the finite difference interpretation of mixed and mixed-hybrid schemes is lost, so that there is no way to obtain, for parabolic problems, computationnally explicit schemes. But the mixed-hybrid formulation still leads to solving, at each time step, a symmetric positive definite linear system for the edge pressure unknowns TP_A, which reduces, when the storage coefficients c_K tend to zero, to the symmetric positive definite linear system governing the elliptic equation.

Notice also that there is no principle objection to the use of general quadrilateral elements. But the application mapping such an element onto the unit square $\hat{K}$ is no longer affine, hence the computation of the elementary matrix A_K is more complicated, and requires a more precise quadrature formula than the Gauss formula commonly used for parallelograms and triangles.

For more details on how to compute the A_K elementary matrix from the basis functions on the reference element K, we refer the reader to references [6].

• Generalization to 3-D irregular meshes is possible using parallelepipedic and tetraedrics elements. But the manipulation of 3-D meshes containing tetreadrons is quite heavy . So maybe 3-D general "quadrilateral" elements would allow for a more easy geometrical description of the physical model to be solved, thus compensating for the burden of tedious calculations of the elementary matrices A_K.

• Generalization to higher order schemes is possible, using higher-order Raviart Thomas or Brezzi-Douglas-Marini spaces for $\overrightarrow{X_K}$. For example, the next Raviart Thomas space would lead to the following set of unknown :

(6.10) - an element pressure P_K which is bilinear (for a rectangular element K with sides parallel to the axes) or linear (for a triangular element K) on K.

(6.11) - an edge pressure $TP_{K,A}$ which is linear on each edge

(6.12) - a flux normal component $\vec{q}_{K,h}\cdot\vec{\nu}_A$ which is linear on each edge

The lowest order Brezzi-Douglas Marini space would have the same order of precision for the velocities as(6.10) through (6.12), but with fewer degrees of freedom for the pressure :

(6.13) - one constant element pressure P_K

(6.14) - an edge pressure $TP_{K,A}$ as in (6.11)

(6.15) - a flux normal component as in (6.12)

We refer to [3] [15] [16] for more details on these high order approximations.

7. Convergence rates and post processing

In this section we make some comments on the rates of convergence of the numerical schemes discussed in the previous sections. As in the earlier paragraph, we suppose that Ω is a rectangle covered by a uniform grid $\mathcal{T}_h$ of square elements of side length h. We consider here only the elliptic problem (4.2) with (4.11) :

$$- \nabla (a(x) \nabla P) = f(x) \qquad \text{in } \Omega$$

and we make the assumption (4.9) concerning the coefficient a(x) :

$$a(x) = a_{min} > 0 \qquad \text{in } \Omega$$

We suppose further that we have a homogeneous Dirichlet boundary condition :

$$P = 0 \qquad \text{on } \Gamma$$

As a point of comparison we first consider the classical Q_1 finite element scheme described in paragraph 3, (3.6). Here the function P_h approximating the pressure P is piecewise bilinear and continuous. It is a classical result cf. [8] theorem 3.2.5 that if a(x) is sufficiently smooth, then [(1)]

$$(7.1) \qquad \| P - P_h \|_{L^2(\Omega)} \leq Ch^2 |P|_{2,\Omega}.$$

This is an optimal order, $o(h^2)$, estimate for approximation in a space of piecewise bilinear functions. One also has, [8] theorem 3.2.2

$$(7.2) \qquad \| P - P_h \|_{H^1(\Omega)} \leq Ch |P|_{2,\Omega},$$

and thus if we define $\vec{q}(x) = - a(x) \nabla P(x)$ and $\vec{q}_h(x) = - a_h(x) \nabla P_h(x)$ it is easy to show that if a is regular enough then

$$(7.3) \qquad \|\vec{q} - \vec{q}_h \|_{L^2(\Omega)} \leq Ch |P|_{2,\Omega}.$$

For the mixed method described in section 4 the approximation P_h to the pressure P is piecewise constant while the velocity $\vec{q}$ is approximated by a vector function $\vec{q}_h$ whose x-component is continuous and piecewise linear in x and piecewise constant in y and whose y-component is continuous and piecewise linear in y and piecewise constant in x. The error made by this approximation is estimated by (cf. [16] theorem 13.2 and theorem II.2)

$$(7.4) \qquad \| P - P_h \|_{L^2(\Omega)} + \| \vec{q} - \vec{q}_h \|_{H(\mathrm{div};\,\Omega)} \leq Ch \left(|P|_{1,\Omega} + |\vec{q}|_{1,\Omega} + |\nabla \vec{q}|_{1,\Omega}\right)$$

---

(1) $$|P|_{m,\Omega} = \left\{ \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_n = m} \left| \frac{\partial^m P}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^2_{L^2(\Omega)} \right\}^{\frac{1}{2}}$$

if $a(x)$ is regular enough for a_h to be an $o(h)$ approximation to a. For $a = a_h$ and h sufficiently small it has been shown, cf.[16], theorem 6.1

$$(7.5) \qquad \| P - P_h \|_{L^2(\Omega)} \leq Ch \|P\|_{H^2(\Omega)}$$

and

$$(7.6) \qquad \| \vec{q} - \vec{q}_h \|_{H(\mathrm{div};\Omega)} \leq Ch \|\vec{q}\|_{H^1(\Omega)}$$

For the velocity this is the same order, $o(h)$ error as for the velocity approximation determined from the Q_1 finite element approximation of P, but the mixed estimation (7.6) involves the stronger norm $(\|\vec{q}\|^2_{L^2} + \|\nabla\vec{q}\|^2_{L^2})^{\frac{1}{2}}$ of H (div,Ω) whereas the Q_1 estimation (7.3) involves only the usual $\|\vec{q}\|_{L^2}$ norm of $L^2(\Omega)$. Morever as pointed out earlier, the continuity of the fluxes for the mixed approximation $\vec{q}_h$ of the velocity makes it a physically much more meaningfull approximation for $\vec{q}$ than would be the approximation obtained a posteriori from the Q_1 finite element procedure.

If one is also interested in an accurate approximation of P one notes that the mixed or mixed hybrid approximation P_h, being piecewise constant, can not give the same accuracy as that obtained from the Q_1 finite element approximation; it is $o(h)$ instead of $o(h^2)$. However one can show that by demanding a bit more regularity for the pressure, H^3 instead of H^2, that one has super convergence (i.e. $o(h^2)$) of P_h to the average value of P, on each element K of $\mathcal{T}_h$:

$$(7.7) \qquad \|P_h - \Pi_h P\|_{L^2(\Omega)} \leq Ch^2 \|P_3\|$$

where Π_h P is the orthogonal L_2-projection of P onto the space of piecewise constant functions over the grid $\mathcal{T}_h$ cf. [16] (6.3).

For the mixed finite element scheme with a quadrature rule described in section 5, one might expect some loss of precision due to the error induced by the numerical integration. However, it is not difficult to show using [16] Theorem II.2 that the order of convergence is preserved. If P_h^* and q_h^* are the approximates to P and q determined in section 5 we have

$$(7.8) \quad \| P - P_h^* \|_{L^2(\Omega)} + \| q - q_h^* \|_{H(\mathrm{div};\Omega)} \leq Ch\, (|P|_{1,\Omega} + |\vec{q}|_{1,\Omega} + |\nabla . \vec{q}|_{1,\Omega}).$$

The mixed hybrid method yields the same approximation P_h to P and approximation $\vec{q}_h$ to $\vec{q}$. as.the mixed method; thus we have the same error estimates (7.4) thu (7.8). However, for the mixed hybrid method we also have an approximation TP_h to the trace TP of P on the edges of the grid. One may also show in the same manner as is done for the case of triangular elements in corollary 1.5 of [1] that if $| \ |_3$ is a suitably defined norm again demanding H^3 regularity of the pressure one has $o(h^2)$ convergence of the trace TP_h to the average value of trace of P, on the edges of the mesh :

$$(7.9) \qquad \|TP_h - \tilde{\Pi}_h P\|_{-\frac{1}{2},h} \leq Ch^2 |P|_3$$

where $\tilde{\Pi}_h P$ is the L_2 projection of TP onto the space of piecewise constant functions on the union of the edges $A \in A_h$ and the norm $\| \ \|_{-1/2,h}$ on this space is given by

$$(7.10) \qquad \|\lambda\|_{-1/2,h} = \left\{ \sum_{\substack{A \in A_h \\ A \text{ interior}}} h \|\lambda\|^2_{L^2(A)} \right\}^{1/2}$$

If one is interested in a more occurate approximation of the pressure than P_h, the trace values TP_h may be used together with the pressure values P_h to obtain a postprocessed non conforming (discontinuous) approximation $\tilde{p}_h$ of the pressure for which one has $o(h^2)$ accuracy :

$$(7.11) \qquad \|\tilde{P}_h - P\|_{L^2(\Omega)} \leq Ch^2 \|P\|_3 . \ q$$

The construction of $\tilde{p}_h$ proceeds as follows : define D(K) for each $K \in \mathcal{T}_h$ by

$$(7.12) \qquad \begin{aligned} D(K) = \{ \varphi|_K : \varphi \in P_2,\ \varphi(x) = ax^2 + by^2 + cx + dy + e, \\ a,b,c,d, \text{ and } e \in \mathbb{R} \}. \end{aligned}$$

Then $\tilde{p}_h|_K$ is the element of D(K) whose integral over K is $P_K h^2$ and whose integral over each edge of K is $TP_{K,A}\,h$. This is easily done by solving small linear systems on each element K.

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